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ASTEROIDAL TRIPLES OF EDGES IN BICHORDAL GRAPH: A COMPLETE LIST

ASHOK KUMAR DAS, RITAPA CHAKRABORTY, MALAY KUMAR SEN

ABSTRACT : Three mutually separable edges of a graph form an *asteroidal triple of edges (ATE)* if for any two of them, there is a path from the vertex set of one to the vertex set of another that avoids the neighbours of the third edge. A bipartite graph is said to be *chordal bipartite* or *bichordal* if it does not contain a chordless cycle of length greater than five. In this paper we provide a complete list of minimal bichordal graphs containing an asteroidal triple of edges.

Key words : asteroidal triple, bichordal, Ferrers digraph, asteroidal triple of edges(ATE)
AMS Classification. 05C62.

1. INTRODUCTION

Three independent vertices in a graph form an asteroidal triple (AT) if for any two of them, there is a path that avoids the neighbours of the third vertex. The motivation to work on this topic came from the study of interval graphs. A graph is an *interval graph* if its vertices can be put in a one-to-one correspondence with a family of intervals on the real line in such a way that two vertices are adjacent if and only if the corresponding intervals intersect. An interval graph has a linear structure in the sense that the set of its maximal cliques has a linear order [5]. Intuitively an asteroidal triple of vertices in graph allows a graph to "grow" in three directions. Proceeding along this line Lekkerkerker and Boland [10] proved that a graph is an interval graph if an only if it is chordal and AT-free. Corneil, Olariu and Stewart [1, 2] have done excellent and extensive work on asteroidal triples. In [1] they have provided a complete list of AT-minimal graphs while in [2] they have studied thoroughly the structural properties of AT-free graphs.

An *interval digraph* is a directed graph such that to every vertex v can be assigned an ordered pair (S_v, T_v) of intervals on the real line so that uv is a directed edge if and only

if S_u interesects T_v. An *interval bigraph* is a bipartite graph such that each vertex v can be assigned an interval on the real line so that two vertices in the opposite partite sets are adjacent if and only if their corresponding intervals interesect. These two models are essentially equivalent and have been studied in [3, 13, 14, 15, 16, 17].

Given a digraph $D(V, E)$, consider the bipartite graph $B = B(D)$ whose partite sets are two disjoint copies U and V of the vertex set V of D and let two vertices u and v in $B(D)$ be adjacent if and only if $uv \in E$. Then it easily follows that D is an interval digraph if the corresponding digraph $B(D)$ is an interval bigraph. The graph $B(D)$ is called the split of D. On the other hand, $B(U, V, E)$ is an interval bigraph if the directed graph $D(U \cup V, E)$ obtained from B by directing all the edges from U to V is an interval digraph. In [14], they were characterized in terms of Ferrers digraphs. A Ferrers digraph satisfies either of the following two equivalent conditions:

i) Successors sets (equivalently predecessor sets) are linearly ordered by inclusion.

ii) The adjacency matrix has no 2×2 permutation matrix as a sub matrix.

The adjacency matrix of a Ferrers diagraph is called a *Ferrers Matrix.*

A binary matrix with exactly one 0 is a Ferrers matrix. Hence every digraph is the intersection of finitely many Ferrers diagraphs. The minimum number of Ferrers digrphs whose intersection is D is the Ferrers dimension of the digraph D, written $fdim(D)$. The bipartite analogue of a Ferrers diagraph is a Ferrers bigraph and $fdim(D)$ is also the Ferrers dimension of the corresponding bigraph.

Analogous to the notion of a containment graph [7, 8], a containment digraph is introduced and studied in [16]. An *interval containment bigraph* or simply a *containment bigraph* is a bipartite graph $B(U, V, E)$ for which there are two families of intervals $\{S_u : u \in U\}$ and $\{T_v : v \in V\}$ such that $u \in U$ is adjacent to $v \in V$ iff S_u contains T_v. An interval bigraph is of Ferrers dimension at most 2; but the converse is not true [14]. A containment bigraph is exactly what characterizes a bigraph having Ferrers dimension at most 2 [16].

A bigraph is *chordal bipartite* or simply *bichordal* [6] if very cycle of length ≥ 6 has

a chord. It is to be noted that every cycle of length ≥ *6* is of Ferrers dimension ≥ 3. Consequently it follows that a bigraph of Ferrers dimension at most *2* is necessarily bichordal.

Motivated by the characteriation of an interval graph in terms of AT's [11] Müllar introduced the interesting notion of an asteroidal triple of edges. A pair of edges x_1y_1 and x_2y_2 of a bipartite graph *H(X, Y, E)* is *separable* [6] if the corresponding vertices induce the subgraph $2K_2$ in *H*. A bigraph containing at least one pair of separable edges is *separable*; otherwise it is *non-separable*. It is clear that a bigraph is a Ferrers bigraph if and only if it is non-separable.

Three mutually separable edges e_1, e_2, e_3 of a bigraph are said to form an *asteroidal triple of edges (ATE)* if for any two of them there is a path joining a vertex of one edge and a vertex of another that avoids the neighbours of the third edge. This is slightly modified version of the definition given by Müllar; we have added the extra condition that the three edges must be mutually separable. In our defination a chordless cycle of length ≥ 10 has an ATE, while a cycle of length ≤ 8 has no ATE.

In [11] Müller had shown that interval bigraphs are ATE free. In [4] it is shown that bigraphs of Ferrers dimension ≤ 2 (which includes interval digraphs) are ATE - free. It is also shown there that the convers is not true. In a step towards studying the structural properties of ATE-free bigraphs, in this paper we start with bichordal graphs and provide a complete list of ATE minimal bichordal graphs in the sense that they contain ATE's and are minimal with this property. In the process we follow the method as adopted in [10].

4. MINIMAL BICHORDAL GRAPHS CONTAINING ATE'S

In this section we determine a minimal set of bichordal graphs with the property that any bichordal have an ATE if and only if it contains a graph of this set as an induced subgraph. A complete list of these bigraphs is given in Fig 1. We have also indicated the three edges which constitute an ATE, in each graph (as bold edges).

Theorem. *A bichordal graph has an ATE if and only if it contains any of the graphs of Fig 1, as an induced subgraph.*

We first prove the following lemmas.

Lemma 1. *Let H (X, Y, E) be a bichordal graph and let $x_1y_1x_2y_2 \ldots\ldots x_ky_kx_1 (k \geq 3)$ be a cycle in H. Then of the four consecutive vertices x_1, y_1, x_2, y_2*

i) x_1y_2 *is a chord in the cycle.*

Or ii) There is a chord incident to any of y_1 and x_2 in the cycle.

Proof : We prove the lemma by induction on k. For k=3, the lemma is obviously true, since H is bichordal. We suppose that the lemma is true for any cycle with number of vertices less than $2k$. Let $C = x_1y_1x_2y_2 \ldots\ldots x_ky_k$ $(K>3)$ be a cycle in H, and let there be no chord incident to either y_1 or x_2. Since H is bichordal, there must be a chord joining a vertex of $\{y_2, y_3, \ldots, y_k\}$ and a vertex of $\{x_3, x_4, \ldots\ldots, x_k, x_1\}$. Thus we get a shorter cycle containing the vertices x_1, y_1 and x_2 having the number of edges less than $2k$. So by hypothesis x_1y_2 is a chord in the shorter cycle and hence is a chord in C.

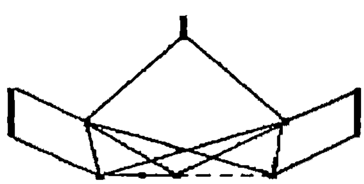
A path $x_1y_1x_2y_2 \ldots\ldots x_ky_k$ in a bichordal graph $H(X, Y, E)$ is *chordless* if it is the induced subgraph of H induced by the vertices $\{x_1, y_1, x_2, y_2 \ldots\ldots x_k, y_k\}$.

Lemma 2. *Let $H(X, Y, E)$ be a bichordal graph, $x_0y_1x_1y_2x_2 \ldots\ldots x_ky_kx_0$ be a cycle and $y_1x_1y_2x_2 \ldots y_k$ a chordless path W in H, then $x_0y_i \in E$ for all y_i, $2 \leq i \leq k-1$.*

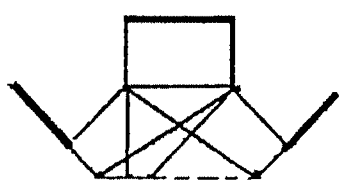
Proof : By lemma 1, with x_0 instead of x_1, we have $x_0y_2 \in E$, since the path $y_1x_1y_2 \ldots\ldots x_ky_k$ is chordless. And repeated application of lemma 1 to shorter cycle proves the lemma.

Now, we are equipped to prove the theorem.

Proof of the Theorem : Necessary part of the theorem is trivial, because each graph of Fig.1 has an ATE. For sufficiency, we have to show that if H is bichordal but not ATE-free then H must



In $(n \geq 9)$



IIn $(n \geq 10)$

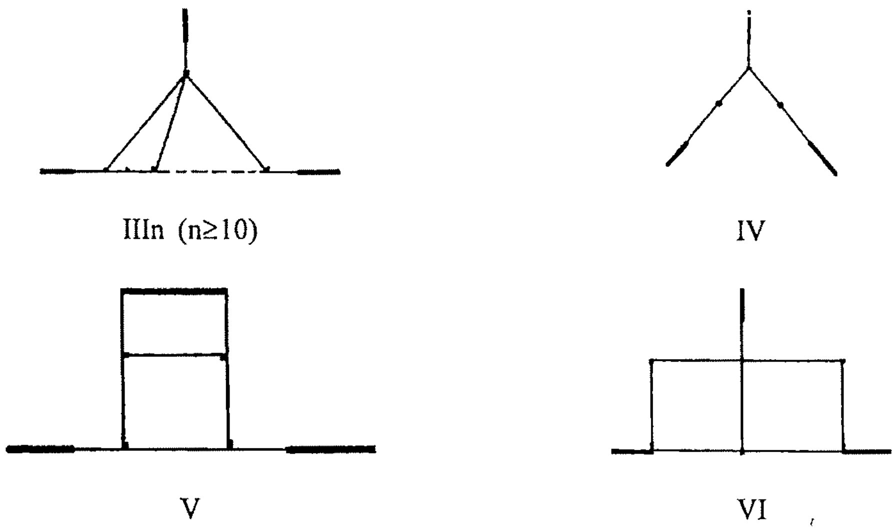


Fig. 1

contain one of the graphs of Fig. 1 as induced subgraph. So the proof of the theorem will be complete if we show that every bipartite graph H having the following properties:

1) H is bichordal,
2) H contains an ATE,
3) H is minimal, i.e., no proper sub-graph has an ATE;

then H is one of the graphs of Fig. 1.

Let a bipartite graph $H(X, Y, E)$ satisfy the properties (1)-(3) and suppose that $\{e_1, e_2, e_3\}$ is an ATE of H. Let W_1, W_2, W_3 be the three paths such that

a) W_i connects two edges e_j $(j \neq i)$

b) e_i has no neighbor in W_i where $i = 1, 2, 3$.

Now we may consider the three paths W_i as chordless and condition (3) implies that H is the union of three paths W_1, W_2 and W_3.

Again, minimality of H implies that each path W_i contains only one neighbor of e_j $(j \neq i)$. So each $e_i = x_iy_i$ has at most two neighbors in H. If y'_i and x'_i are respectively the neighbours of x_i and y_i, then lemma 1 implies $x'_iy'_i \in E$.

Now we have to consider the following cases:

Case 1 : Each $e_i (i = 1, 2, 3)$ has exactly two neighbors.

Case 2 : One of the edges has exactly one neighbor.

First, we study case 1. Here we have four sub-cases.

Case 1.1: Each e_i has two neighbours incident to two different vertices of e_i.

Case 1.2: One $e \in \{e_1, e_2, e_3\}$ has two neighbours adjacent to same endpoint of e and the two edges in $\{e_1, e_2, e_3\} \setminus \{e\}$ have two neighbours incident to different vertices of $\{e_1, e_2, e_3\} \setminus \{e\}$.

Case 1.3: Two edges in $\{e_1, e_2, e_3\}$ have two neighbours incident to same vertex and the remaining edge has two neighbours incident to different vertices.

Case 1.4 : Each $e_i (i = 1, 2, 3)$ has two neighbours incident to same vertex of e_i.

First we consider the case 1.2. In this case we have two possibilities;

i) both ends of all W_i are in the same partite set of vertex, and

ii) both ends of one W_i are in same partite set of the vertex and two ends of other two W_i are in opposite partite set.

In case 1.2.i suppose each W_i is given by

$W_1 = x_2 y'_2 \ldots y_3 x_3$; $W_2 = y_1 x_1'' \ldots x'_3 y_3$ *and* $W_3 = y_1 x'_1 \ldots x'_2 y_2$. We must not exclude the case that $x'_1 = x'_2$, $x''_1 = x_3'$, $y'_2 = y'_3$.

And in this case we say the corresponding path is *short*.

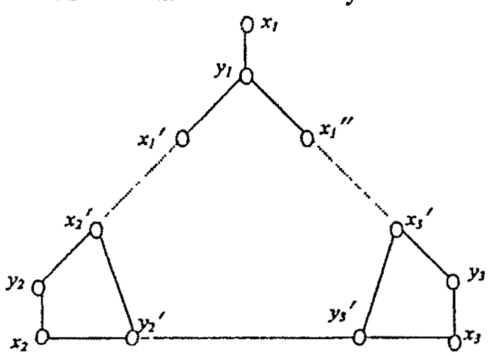


Fig. 2.1

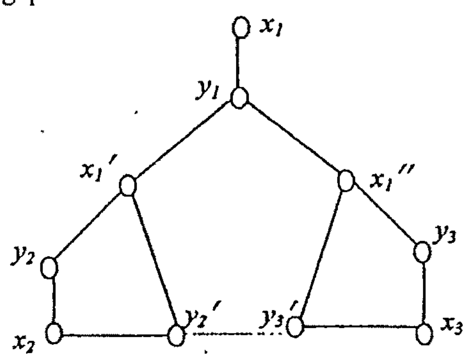


Fig. 2.2

So we have the situation of Fig. 2.1. Here the paths W_i may have interior vertices in common.

Now we shall show that the paths W_2 and W_3 are short. If these two parts are not short then applying lemma 1 to the cycle

$y_1x'_1 \dots x'_2y'_2 \dots y'_3x'_3 \dots x''_1y_1$ we see that there always exists a sub-graph of the graph H of Fig. 2.1 which contains an *ATE*, i.e., H is not minimal. So, this contradiction implies that W_2 and W_3 are *short.*

Thus we arrive at the situation of Fig. 2.2. Now, if $x''_1y'_2 \notin E$ then the three edges e_1, e_3 and $x_2y'_2$ form an *ATE* of $H/\{y_2\}$. So minimality of H implies that $x''_1y'_2 \in E$. Again applying lemma 2 to cycle $x''_1y'_2 \dots y'_3x''_1$ we have $x''_1y \in E$ for all $y \in W_1$. Similarly we can show that $x'_1y \in$ E for all $y \in W_1$. Then H is form I_n.

From the remaining possibility we can check that no new minimal graph containing an *ATE* can be found.

Case 1.3 In this case we also have two possibilities (*a*) two end vertices of two W_i are in same partite set of vertex set and (b) two ends of all W_i are in the opposite partite set of vertex set.

In the first case (Fig. 3.1) let each path W_i is given by

$W_1 = x_2y'_2 \dots \dots x'_3y_3$, $W_2 = y_1x'_1 \dots \dots x''_3y_3$, $W_3 = x_1y'_1 \dots\dots y''_2x_2$.

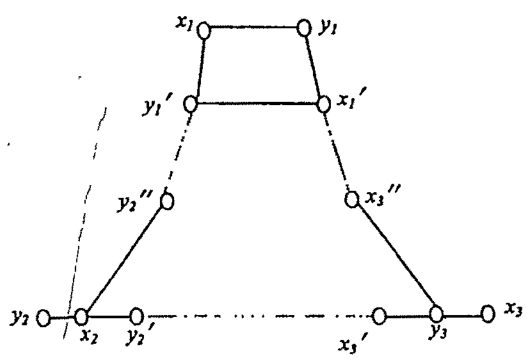


Fig 3.1

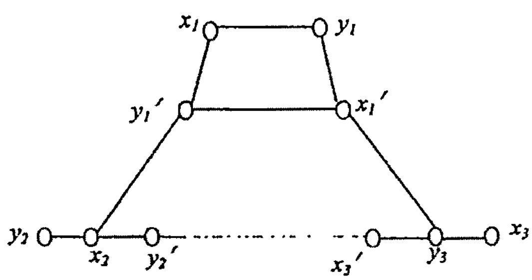


Fig. 3.2

Here $y'_2 \neq y''_2$ and also $x'_3 \neq x''_3$. But it may happen that $y'_1 = y''_2$ and $x'_1 = x''_3$ and in that case we say the corresponding path is short. (The path W_1 will be called short if $x'_3y'_2 \in E$).

Now as before we can show that the paths W_2 and W_3 are short. Since otherwise applying lemma 1 to the cycle $x_2y'_2 \ldots x'_3y_3x''_3 \ldots \ldots x'_1y'_1 \ldots y''_2x_2$ we find that there always exist a subgraph of Fig. 3.1 containing an ATE. Thus we have the situation of Fig. 3.2.

Observe that $x'_3y'_1$ must belong to E, since otherwise the set $\{e_3, e_2, x'_3y'_1\}$ of edges is an *ATE* of $H/\{x_3\}$. And then by lemma 2, $y'_1x \in E$ for all $x \in W_1$ similarly we find that $x'_1y \in E$ for all $y \in W_1$. So H is of the form II_n.

Similarly, in the other cases it can be verified that no new minimal graph containing an ATE will be found.

Case 1.1 in this case also we have two possibilities (a) two ends of all W_i's belongs to opposite partite set of vertices and two ends of the remaining W_i belong two opposite parity of vertices.

In case 1.1.a suppose each W_i be given by

$Wi = x_3y'_3 \ldots x'_2y_2$, $W2 = x_1y_1 \ldots \ldots x'_3y_3$, $W_3 = x_2y'_2 \ldots \ldots x'_1y_1$.

Here we must not exclude the possibility that $x'_2y'_3$, $x'_3y'_1$, $x'_1y'_2$ may belong to E. And in this case we say that the corresponding path is short.

Now, the three vertices x'_1, x'_2 and x'_3 are pairwise different (as e_i as no neighbour in W_i). And this is also true for the vertices y'_1, y'_2, y'_3. So we have the situation of Fig. 4.1. But the paths W_i ($i = 1, 2, 3$) may have interior vertices in common.

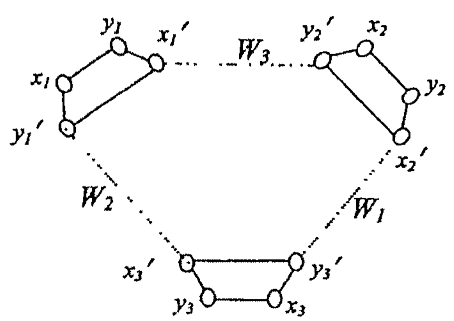


Fig. 4.1

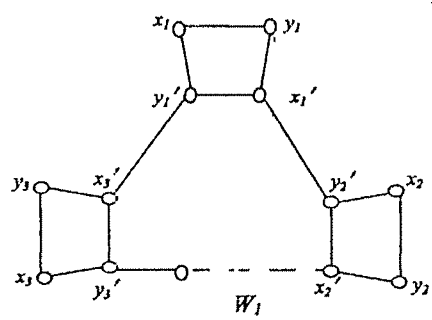


Fig. 4.2

As before we can prove that at least two paths W_i, $i = 1, 2, 3$ are short. Let the paths W_2 and W_3 be short. Thus we have the situation of Fig. 4.2

Now, if $x'_1y'_3 \in E$, then $\{e_1, e_2, x_3y'_3\}$ is an ATE of $H/\{y_3\}$. So minimality of H implies that $x_1y'_3 \in E$. Similarly we have $x'_2y'_1 \in E$ and $x'_3y'_2 \in E$.

Again if W_1 is not short then $\{e_1, e_2, x_3y'_3\}$ is an ATE of $H / \{y_3\}$. So, W_i must be short and consequently H is of the form of Fig. 4.3.

But the graph H of the Fig. 4.3 is not minimal. Since $H / \{x'_2, y'_3\}$ contains an ATE. Thus we have no graph satisfying properties (1), (2) and (3) in this case.

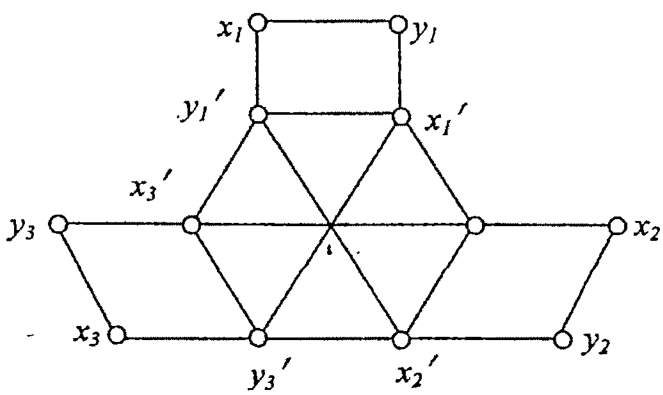


Fig. 4.3

Case 1.4. We have similar conclusion in this case also.

Case 2. Let $e_1 = x_1y_1$ has only neighbour, say y'_1. If y'_1 has no neighbour in W_1 then $\{e_1, e_2$ and $x_1y'_1\}$ is an ATE of $H / \{y_1\}$. So, y'_1 must have some neighbour in W_1. Now we distinguish the following sub-cases.

Case 2.1. y'_1 has $k \geq 2$ neighbours in W_1. Let x'_1 and x'' be the first and last neighbour of $y'_{.1}$ in the path v_2 ... ...x' ... ...x'' ... v_3 (v_2 and v_3 are respectively any vertex of e_2 and e_3). Becasue W_1 is irreducible so the portion x' ... x'' of W_1 is also irreducible and we can apply lemma 2. Again y'_1 is not a neighbour of e_2 and e_3. So it follows that H contains a graph IIIn.

Case 2.2. y'_1 has only one neighbour $x_0 \in W_1$. Then necessarily $x_0 \in W_1$, $x_0 \in W_2$, and $x_0 \in W_3$. Also x_0 is not a neighbour of e_2 and e_3. So, H contains the graph IV.

Case 2.3. y'_1 has only one neighbour $x_0 \in W_1$ and also has only one neighbour x'_1 other than x_0 and x_1 and say, $x'_1 \in W_3$.

Here we have two possibilities (i) e_2 has two neighbours v'_2 and v''_2 incident to different vertices of e_2 (ii) e_2 has two neighbours incident two same vertex same vertex of e_2.

We write

$W_3 = v_2v'_2 \ldots \ldots x'_1y'_1x_1$; $\quad W_1 = v_2v''_2 \ldots \ldots y'x_0 \ldots \ldots v'_3v_3$ and

$W_2 = x_1y'_1x_0y'' \ldots \ldots v'_3v_3$.

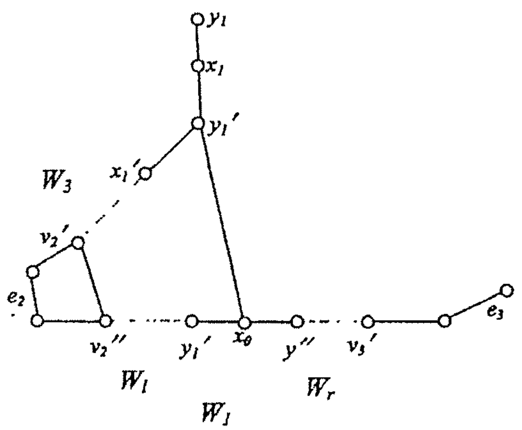


Fig. 5.1

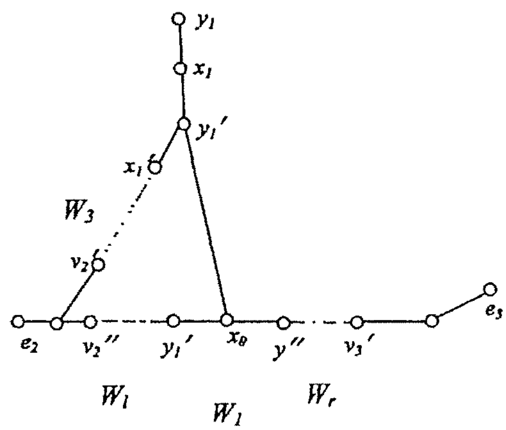


Fig. 5.2

We denote the portions $v''_2 \ldots \ldots y'$ and $y'' \ldots \ldots v'_3$ of W_1 by W_l and W_r respectively.

First we shall show that the portion $v'_2 \ldots x'_1$ of W_3 does not contain a vertex v such that $vv' \in E$ for any vertex $v' \in W_r$. If possible let such v exist; then we can replace W_1 and W_2 respectively by

$W'_1 = v_2v'_2 \ldots \ldots vv' \ldots \ldots v'_3v_3$ (does not contain x_0)

$W'_2 = x_1y'_1x'_1 \ldots \ldots vv' \ldots \ldots v'_3v_3$ (does not contain x_0)

Then we can easily check that $\{e_1, e_2, e_3\}$ is an ATE of $H / \{x_0\}$ (actually $H / \{x_0\}$ contains the graph IV). Thus by the minimality of H we must have $vv' \notin E$.

Now, we apply lemma 1 to the cycle $y'_1x'_1 \ldots \ldots v'_2 \ldots v''_2 \ldots y'x_0y'_1$.

Since y'_1 has no neighbour in W_1 (other than x_0) so x_0 or y' must have neighbours W_3. If x_0 have neighbours in W_3 then H must contain the graph IIIn i.e., we have no new graph containing an ATE in this case. But if x_0 has no neighbour in W_3 but y' has one neighbour, say, x in W_3 then by repeated application of the lemma 1 to the shorter cycles $y'_1x_0y'x \ldots \ldots x'_1y'_1$ we find that the graph H contains the structure V.

Case 2.4 y'_1 has two neighours x' and x'' other than $x_0 \in W_1$ and x_1, where $x' \in W_3$ and $x'' \in W_2$. Here we suppose that both e_2 and e_3 have two neighbours incident to the same vertex of $e_2(e_3)$.

In this case similarly we can prove that if $v \in W_3$ and $v' \in W_r$ then $vv' \notin E$ and if $v \in W_2$ and $v'' \in W_1$ then also $vv'' \in E$. Also if $v'' \in W_2$ $v'''\in W_3$ *then* $v''v''' \notin E$. Since if $v''v''' \in E$ then we can replace W_1 by $W'_1 = v_2 \ldots \ldots v''v''' \ldots v_3$ which does not contain x_0. So $\{e_1, e_2, e_3\}$ is an ATE of $H/\{x_0\}$ which contradicts the minimality of H.

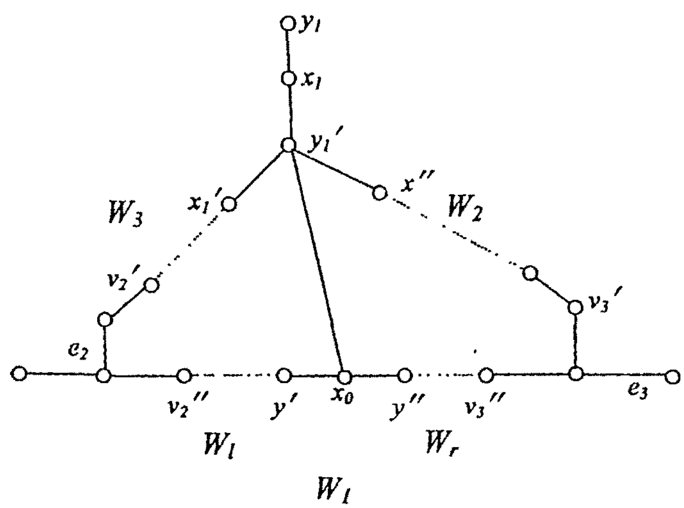


Fig. 6

Next applying lemma 1 to the cycles.

$y'_1x_0y' \ldots \ldots v''_2v_2v'_2 \ldots x'y'_1$ and $y'_1x_0y'' \ldots v''_3v_3v'_3 \ldots \ldots x''y_1$ as in the previous case we find that H contains the graph VI.

In the other possibilities also we can find that H contains the graph VI.

This completes the proof of the theorem.

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SOME RESULTS IN A KÄHLER MANIFOLD EQUIPPED WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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ABSTRACT : In the present paper we have studied unit parallel vector field and Kiling vector field in a Kähler manifold equipped with semi-symmetric non-metric connection. We have also obtained certain results for curvature tensor and Lie derivative with respect semi-symmetric non-metric connection.

2000 Mathematics Subject Classification : **53C15**.

Key words : Semi-symmetric non-metric connection, Kähler manifold, Unit parallel vector field, Killing vector field.

1. INTRODUCTION

Cengizhan Murathan and Cihan Özgur [1]discussed the Riemannian manifolds with a semi-symmetric metric connection satisfying some semi-symmetry conditions. Also the Riemannian manifold equipped with a semi-symmetric metric connection have been studied by O.C. andonic [3], M.C. Chaki and A. Konar [4], U.C. De [5] etc., while a special type of semi-symmetric metric connection on a weakly symmetric Riemannian manifold has been studied by U.C. De and Joydeep Sengupta [6]. A Kähler manifold equipped with semi-symmetric metric connection, a semi-symmetric non-metric connection on a Kähler manifold and an almost Hermitian manifold with semi-symmetric recurrent connection have been studied by P.N. Pandey and B.B. Chaturvedi [7, 8, 9]. Nirmala S. Agashe and Mangala R. Chafle [2] have studieds emi-symmetric non-metric connection on a Riemannian manifold in 1992. Let M_n be an even dimensional differentiable manifold of differentiability class C^{r+1}. If there exists a vector valued linear function F of differentiability class C^r such that for any vector field X.

(1.1) $\overline{\overline{X}} + X = 0.$

(1.2) $g\left(\overline{X}, \overline{Y}\right) = g(X, Y),$

and

(1.3) $(D_X F)Y = 0.$

where $\overline{X} = FX$, g is non-singular metric tensor and D is the Riemannian connection, then M_n is called a Kähler manifold. We define another linear connection ∇ for two arbitrary vector fields X and Y such that

(1.4) $\nabla_X Y = D_X Y + a\omega(X)Y + b\omega(Y)X.$

where ω is a 1-form associated with an unit vector field ρ which is parallel with respect to Ricmannian connection D and defined by $\omega(X) = g(X, \rho)$. Here a and b are non-zero real or complex numbers such that $a \neq b$.

Putting $g(Y, Z)$ in place of Y in (1.4), we have

(1.5) $(\nabla_X g)(Y, Z) = -a\omega(X)g(Y, Z) - b\omega(Y)g(X, Z) - b\omega(Z)g(X, Y).$

which shows that the connection ∇ is non-metric.

2. PARALLEL UNIT VECTOR FIELD

A vector field U is said to be a parallel unit vector field with respect to the Riemannian connection D if

(2.1) $DU = 0$, *and* $||U|| = 1$.

Putting $Y = U$ in equation (1.4), we have

(2.2) $\nabla_X U = D_X U + a\omega(X)U + b\omega(U)X.$

Using (2.1) in (2.2), we have

(2.3) $\nabla_X U = a\omega(X)U + bX.$

Using $\omega(X) = g(X, U)$ in equation (2.3), we can write

(2.4) $$\omega(\nabla_X U) = a\omega(X)\omega(U) + b\omega(X).$$

Again using (2.1), we have

(2.5) $$\omega(\nabla_X U) = (a + b)\omega(X).$$

This implies

(2.6) $$\nabla_X U = (a + b)X.$$

Now, we conclude:

Theorem 2.1. *In a Kähler manifold equipped with semi-symmetric non-metric connection ∇, if U be a unit parallel vector field with respect to Riemannian connection D then it will be parallel vector field with respect to semi-symmetric non-metric connection ∇ if and only if $a = -b$.*

Putting $Z = U$ in equation (1.5) and using (2.1), we have

(2.7) $$(\nabla_X g)(Y, U) = -(a + b)\omega(X) - bg(X, Y).$$

Therefore, we can state that :

Theorem 2.2. *In a Kähler manifold equipped with semi-symmetric non-metric connection ∇, if U is a unit parallel vector field with respect to Riemannian connection D then $(\nabla_X g)(Y, U)$ is symmetrical in X and Y.*

3. CURVATURE TENSOR

From (1.4), we have

(3.1) $$\nabla_Y Z = D_Y Z + a\omega(Y)Z + b\omega(Z)Y.$$

Replacing Y for $\nabla_Y Z$ in equation (1.4), we have

(3.2) $$\nabla_X \nabla_Y Z = D_X \nabla_Y Z + a\omega(X)\nabla_Y Z + b\omega(\nabla_Y Z)X.$$

Using (1.4) In (3.2), we have

(3.3) $$\nabla_X \nabla_Y Z = D_X D_Y Z + a(D_X\omega)(Y)Z + a\omega(D_XY)Z + a\omega(Y)D_XZ$$
$$+ b(D_X\omega)(Z)Y + b\omega(D_XZ)Y + b\omega(Z)D_XY$$
$$+ a\omega(X)D_YZ + a^2\omega(X)\omega(Y)Z + ab\omega(X)\omega(Z)Y$$
$$+ b\omega(D_YZ)X + ab\omega(Y)\omega(Z)X + b^2\omega(Y)\omega(Z)X.$$

Interchanging X and Y in the above equation, we get

(3.4) $$\nabla_Y \nabla_X Z = D_Y D_X Z + a(D_Y\omega)(X)Z + a\omega(D_YX)Z + a\omega(X)D_YZ$$
$$+ b(D_Y\omega)(Z)X + b\omega(D_YZ)X + b\omega(Z)D_YX$$
$$+ a\omega(Y)D_XZ + a^2\omega(Y)\omega(X)Z + ab\omega(Y)\omega(Z)X$$
$$+ b\omega(D_XZ)Y + ab\omega(X)\omega(Z)Y + b^2\omega(X)\omega(Z)Y.$$

From equation (1.4), we may write

(3.5) $$\nabla_{[X,\ Y]}Z = D_{[X,\ Y]}Z + a\omega([X,\ Y])Z + b\omega(Z)[X,\ Y].$$

Subtracting (3.4) and (3.5) from (3.3), we have

(3.6) $$\tilde{R}(X,\ Y)Z = R(X,\ Y)Z + b^2[\omega(X)Y - \omega(Y)X]\omega(Z).$$

Since U is unit parallel vector field with respect to Riemannian connection therefore

(3.7) $$R(X,\ Y)U = 0.$$

Now putting $Z = U$, in (3.6) and using (3.7) and (2.1), we have

(3.8) $$\tilde{R}(X,\ Y)U = b^2[\omega(X)Y - \omega(Y)X],$$

which implies

(3.9) $$\tilde{R}(X,\ Y)U = \tilde{R}(Y,\ X)U,$$

which conclude:

Theorem 3.1. *In a Kähler manifold equipped with a semi-symmetric non-metric connection* ∇, *if U is a unit parallel vector field with respect to Riemannian connection D then*

(i) $\tilde{R}(X, Y)U = \tilde{R}(Y, X)U$

(ii) $\tilde{R}(X, Y)U = 0$ *if and only if* $\omega(X)Y = \omega(Y)X$.

4. KILLING VECTOR

A vector field V is said to be a Killing vector field if Lie derivative of the metric g with respect to V vanishes, i.e.,

(4.1) $$L_V g = 0.$$

Now Lie derivative of Riemannian metric $*(X, Y)$ with respect to semisymmetric non-metric connection ∇ is given by

(4.2) $$(\tilde{L}_V g)(X, Y) = \tilde{L}_V g(X, Y) - g(\tilde{L}_V X, Y) - g(X, \tilde{L}_V Y).$$

Equation (4.1), implies

(4.3) $$(\tilde{L}_V g)(X, Y) = \tilde{\nabla} g(X, Y) - g(\widetilde{[V, X]}, Y) - g(X, \widetilde{[V, Y]}.$$

From the above equation we have

(4.4) $$(\tilde{L}_V g)(X, Y) = \tilde{\nabla} g(X, Y) - g([\nabla_V X - \nabla_X V], Y) + g(X, [\nabla_V Y - \nabla_Y V]).$$

Which implies

(4.5) $$(\tilde{L}_V g)(X, Y) = \tilde{\nabla} g(X, Y) - g(\nabla_V X, Y) + g(\nabla_X V, Y) + g(X, \nabla_V Y). - g(X, \nabla_Y V).$$

Now we know that

(4.6) $$\tilde{\nabla} g(X, Y) - g(\nabla_V X, Y) - g(X, \nabla_Y V) = 0.$$

Using (46) in (4.5), we get

(4.7) $$(\tilde{L}_V g)(X, Y) = (\nabla_V g)(X, Y) + g(\nabla_X V, Y) + g(X, \nabla_V Y).$$

Using (1.4) and (1.5) in (4.7) gives

(4.8) $$\begin{aligned}(\tilde{L}_V g)(X, Y) &= -a\omega(V)g(X, Y) - b\omega(X)g(V, Y)\\ &\quad - b\omega(Y)g(V, X) + g(D_X V + a\omega(X)V\\ &\quad + b\omega(V)X, Y) + g(X, D_Y V + a\omega(Y)V + b\omega(V)Y).\end{aligned}$$

Equation (4.8) implies

(4.9) $$\begin{aligned}(\tilde{L}_V g)(X, Y) &= -a\omega(V)g(X, Y) - b\omega(X)g(V, Y)\\ &\quad - b\omega(Y)g(V, X) + a\omega(X)g(V, Y)\\ &\quad + g(D_X V, Y) + b\omega(V)g(X, Y)\\ &\quad + g(X, D_Y V) + a\omega(Y)g(X, V) + b\omega(V)Vg(X, Y).\end{aligned}$$

Similarly Lie derivative of Riemannian metric $g(X, Y)$ with respect to Riemannian connection D is given by

(4.10) $$(L_V g)(X, Y) = g(D_X V, Y) + g(X, D_Y V).$$

Subtracting equation (4.10) from (4.9), we get

(4.11) $$\begin{aligned}(\tilde{L}_V g)(X, Y) - (L_V g)(X, Y) &= -a\omega(V)g(X, Y) - b\omega(X)g(V, Y)\\ &\quad - b\omega(Y)g(V, X) + a\omega(X)g(V, Y)\\ &\quad + b\omega(V)g(X, Y) + a\omega(Y)g(X, V)\\ &\quad + b\omega(V)g(X, Y).\end{aligned}$$

Putting U in the place of V in equation (4.11) and using (2.1), we get

(4.12) $$(\tilde{L}_U g)(X, Y) - (L_U g)(X, Y) = 2(a - b)\omega(X)\omega(Y) + (2b - a)g(X, Y)$$

Thus, we conclude:

Theorem 4.1. *If V be a killing vector field with respect to the Riemanian connection D in a Kähler manifold equipped with semi-symmetric non-metric connection ∇, then V is a Killing vactor field with respect to the semi-symmetric non-metric connection ∇ if and only if*

$$2(a - b)\omega(X)\omega(Y) + (2b - a)*(X, Y) = 0. \tag{4.13}$$

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FIXED POINT THEOREM FOR ASYMPTOTICALLY REGULAR MAPS IN 2-BANACH SPACE

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ABSTRACT : The object of this paper is to obtain unique fixed point theorems for asymptotically regular maps in 2-Banach Space using generalized contractive condition.

2000 *Mathematics Subject Classification* 47H10, 54H25.

Key words and Phrases : Asymptotically regular maps, 2-normed space, 2-Banach space, fixed point.

1. INTRODUCTION

Considerable attention has been given to fixed points and fixed point theorems in metric and Banach space due to their tremendous applications to mathematics. Motivated by this work, several authors introduced similar concepts and proved analogues fixed point theorems in 2-metric and 2-Banach space as cited in the papers of the following authors. Gahler ([2], [3]) investigated the idea of 2-metric and 2-Banach spaces and proved some results. Subsequently, several authors including Iseki [5], Rhoades [7], and white [9] studied various aspects of the fixed point theory and proved fixed point theorems in 2-metric spaces and 2-Banach spaces. On the other hand, Cho et al. [1] investigated common fixed points of weakly computing mappings and examined the asymptotic regular property in 2-metric space. Panja and Baisnab [6] studied asymptotically regular and common fixed point theorems Saha et al. [8] proved fixed point theorem for mixed contraction using asymptotically regular property in a 2-Banach space.

In spite of the above work, the asymptotic regularity and fixed point theorems on a 2-Banach space need more investigation. So the major objective of this paper is to study some fixed point theorems for generalized contraction mappings from Gopal and Ranadive [4] possessing the asymptotically regular property in a 2-Banach space.

2. PRELIMINARIES

2.1. Definition. Let X be a real linear space and $\|.,.\|$ be a non-negative real valued function defined on $X \times X$ satisfying the following conditions:

i) $\|x, y\| = 0$ *if and only if* x *and* y *are linearly dependent in* X,

ii) $\|x, y\| = \|y, x\|$ *for all* $x, y \in X$

iii) $\|x, ay\| = |a| \, \|x, y\|$ *a being real,* $x, y \in X$

iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ *for all* $x, y, z \in X$

Then $\|.,.\|$ is called a 2-norm and the pair $(X, \|.,.\|)$ is called a linear 2-normed space

Some of the basic properties of 2-norms are that they are non negative satisfying $\|x, y + ax\| = \|x, y\|$, for all $x, y \in X$ and all real numbers a.

2.2 Definition. A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|.,.\|)$ *is called Cauchy sequence if*

$$\lim_{m,n\to\infty} \|x_m - x_n, y\| = 0 \text{ for all } y \text{ in } X.$$

2.3. Definition. A sequence $\{x_n\}$ in a linear 2-normed space $(X, \|.,.\|)$ is said to be convergent if there is a point x in X such that

$$\lim_{n\to\infty} \|x_n - x, y\| = 0 \text{ for all } y \text{ in } X.$$

If $\{x_n\}$ converges to x, we write $\{x_n\} \to x$ as $n \to \infty$.

2.4. Definition. A liner 2-normed space X is said to be complete if every Cauchy sequence is convergent to an element of X. We then call X to be a 2-Banach space.

2.5. Definition. Let X be a 2-Banach space and T be a self-mapping of X. T is said to be continuous at x if for every sequence $\{x_n\}$ in X, $\{x_n\} \to x$ as $n \to \infty$ implies $\{T(x_n)\} \to T(x)$ as $n \to \infty$.

2.6. Example. Let X is R^3 and consider the following 2-norm on X as

$$\|x, y\| = \left|\det\begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}\right|$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then $(X, \| .,. \|)$ is a 2-Banach space.

2.7. Example. Let P_n denotes the set of all real polynomials of degree $\leq n$, on the interval $[0, 1]$. By considering usual addition and scalar multplication, P_n is a linear vector space over the reals. Let $\{x_0, x_1, \ldots\ldots, x_{2n}\}$ be distinct fixed points in $[0, 1]$ and define the following 2-norm on P_n : $\|f, g\| = \sum_{k=0}^{2n} |f(x_k)g(x_k)|$, whenever f and g are linearly independent and $\| f, g \| = 0$, if f, g are linearly dependent. Then $(P_n, \| .,. \|)$ is a 2-Banach space.

2.8. Example. Let X is Q^3, the field of rational number and consider the following 2-norm on X as:

$$\| x, y\| = \left|\det\begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}\right|$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then $(X, \| .,. \|)$ is not a 2-Banach space but a 2-normed space.

2.9. Definition. (Asymptotic regularity in 2-normed linear space) Let $(X, \|.,.\|)$ be a 2-normed linear space with 2-norm $\|.,.\|$. A mapping T of X into itself is said to be asymptoticaly regular at some point x in X if

$$\lim_{n\to\infty} \| T^n(x) - T^{n+1}(x), a \| = 0$$

For all $a \in X$, where $T^n(x)$ denotes the n-th iterate of T at x.

The result of Dhananjay Gopal and A. S Ranadive [4] is given below

Theorem 2.1. Let (X, d) be a complete metric space and T a self map satisfying the inequality:

$$d(Tx,\ Ty) \le \alpha\ \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{[d(x,Ty)+d(y,Tx)]}{2}\right\}$$

for all x, y in X, where $\alpha \in (0, 1)$. If there exist an asymptotically T-regular sequence in X, then T has a unique fixed point.

3. MAIN RESULTS

3.1. Theorem. Let $(X, \|\ .,.\ \|)$ be a 2-Banach space and T be a mapping of X into itself such that for every x, y, $a \in X$ satisfying

$$\|T(x)-T(y), a\| \le \alpha\ \max\left\{\|x-y, a\|, \|x-T(x), a\|, \|y-T(y), a\|, \frac{[\|x-T(y), a\|+\|y-T(x), a\|]}{2}\right\} \quad (3.1.1)$$

where $\alpha \in (0, 1)$. If T is asymptotically regular at some point in X, then T has a unique fixed point in X.

Proof: Let T be asymptotically regular at $x_0 \in X$. Then for positive integers m, n

$$\|T^m(x_0) - T^n(x_0), a\| = \|T(T^{m-1}(x_0)) - T(T^{n-1}(x_0)), a\|$$

$$\le \alpha \max\left\{ \|T^{m-1}(x_0)-T^{n-1}(x_0), a\|, \|T^{m-1}(x_0)-T^m(x_0), a\|, \|T^{n-1}(x_0)-T^n(x_0), a\|, \right.$$

$$\left. \frac{[\|T^{m-1}(x_0)-T^n(x_0), a\|+\|T^{n-1}(x_0)-T^m(x_0), a\|]}{2}\right\}$$

$$\le \alpha \max\left\{ \|T^{m-1}(x_0)-T^m(x_0), a\|+\|T^m(x_0)-T^n(x_0), a\|+\|T^n(x_0)-T^{n-1}(x_0), a\| \right.$$

$$\|T^{m-1}(x_0)-T^m(x_0), a\|, \|T^{n-1}(x_0)-T^n(x_0), a\|,$$

$$\left.\frac{\left[\left\|T^{m-1}(x_0)-T^m(x_0), a\right\|+\left\|T^m(x_0)-T^n(x_0), a\right\|+\left\|T^{n-1}(x_0)-T^n(x_0), a\right\|+\left\|T^n(x_0)-T^m(x_0), a\right\|\right]}{2}\right\}$$

$$\leq a\left\{\left\|T^{m-1}(x_0)-T^m(x_0), a\right\|+\left\|T^m(x_0)-T^n(x_0), a\right\|+\left\|T^n(x_0)-T^{n-1}(x_0), a\right\|\right\}$$

$$\Rightarrow (1-\alpha)\left\|T^m(x_0)-T^n(x_0), a\right\| \leq \alpha\left\{\left\|T^{m-1}(x_0)-T^m(x_0), a\right\|+\left\|T^n(x_0)-T^{n-1}(x_0), a\right\|\right\}$$

$$\Rightarrow \left\|T^m(x_0)-T^n(x_0), a\right\| \leq \frac{\alpha}{1-\alpha}\left\{\left\|T^{m-1}(x_0)-T^m(x_0), a\right\|+\left\|T^n(x_0)-T^{n-1}(x_0), a\right\|\right\}$$

where $\alpha < 1$.

Which tends to 0 as $m, n \to \infty$, since T is asymptotically regular in X. Then $\{T^n(x_0)\}$ is a Cauchy sequence. Since X is a 2-Banach space,

$$\lim_{n\to\infty} T^n(x_0) = u \in X$$

Then

$$\| u-T(u), a \| \leq \left\| u-T^n(x_0), a \right\| + \left\| T^n(x_0)-T(u), a \right\|$$

$$\leq \left\| u-T^n(x_0), a\right\| + \alpha \max\left\{ \left\|T^{n-1}(x_0)-u, a\right\|, \left\|T^{n-1}(x_0)-T^n(x_0), a\right\|, \left\|u-T(u), a\right\|, \right.$$

$$\left.\frac{\left[\left\|T^{n-1}(x_0)-T(u), a\right\|+\left\|u-T^n(x_0), a\right\|\right]}{2}\right\}$$

Letting limit $n \to \infty$ we get

$$\|u - T(u), a\| \leq \alpha \|u - T(u), a\|$$

$$\Rightarrow (1 - \alpha) \|u - T(u), a \| \leq 0$$

$$\Rightarrow \|u - T(u), a\| = 0 \text{ (As } \alpha < 1)$$

$$\Rightarrow u = T(u)$$

Uniqueness : Let $u \neq v$ with $T(v) = v$ for $v \in X$

Then

$$\begin{aligned}\|u - v, a\| &= \|T(u) - T(v), a\| \\ &\leq \|u - v, a\| + \alpha \max \left\{ \|u-v, a\|, \|u-T(u), a\|, \|v-T(v), a\|, \frac{[\|u-T(v), a\| + \|v-T(u), a\|]}{2} \right\} \\ &\Rightarrow \|u - v, a\| \leq \alpha \|u - v, a\| \\ &\Rightarrow (1 - \alpha)\|u - v, a\| \leq 0 \\ &\Rightarrow \|u - v, a\| = 0 \text{ (As } \alpha < 1)\end{aligned}$$

gives a contradiction. Hence $u = v$

This completes the proof of the theorem 3.1.

3.2. Theorem. Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and T be a mapping from X into itself satisfying (3.1.1). If T is asymptotically regular at some point in $x \in X$ and the sequence of iterates $\{T^n(x)\}$ has a subsequence converging to a point $z \in X$, then z is the unique fixed point of T and $\{T^n(x)\}$ also converages to z.

Proof. Let

$$\lim_{k\to\infty} T^{n_k}(x) = z$$

Then

$$\|z - T(z), a\| = \|z - T^{n_k+1}(x), a\| + \|T^{n_k+1}(x) - T(z), a\|$$

$$\leq \|z - T^{n_k+1}(x), a\| + \alpha \max \left\{ \|T^{n_k}(x) - z, a\|, \|T^{n_k}(x) - T^{n_k+1}(x), a\|, \|z - T(z), a\|, \frac{[\|T^{n_k}(n) - T(z), a\| + \|z - T^{n_k+1}(x), a\|]}{2} \right\}$$

Letting $k \to \infty$ we get

$$\|z - T(z),\ a\| \le \alpha\ \|z - T(z),\ a\|$$

$$\Rightarrow (1 - \alpha)\|z - T(z),\ a\| \le 0$$

$$\Rightarrow \|z - T(z),\ a\| = 0 \quad (\text{As } \alpha < 1)$$

$$\Rightarrow z = T(z)$$

Uniquencess of z follows very immediate.

Next

$$\|z - T^n(x),\ a\| = \|T(z) - T^n(x),\ a\|$$

$$\le \alpha \max\left\{\left\|z - T^{n-1}(x), a\right\|, \left\|z - T(z), a\right\|, \left\|T^{n-1}(x) - T^n(x), a\right\|,\ \frac{\left[\left\|z - T^n(x), a\right\| + \left\|T^{n-1}(x) - T(z), a\right\|\right]}{2}\right\}$$

$$\le \alpha \max\left\{\left\|z - T^n(x), a\right\| + \left\|T^n(x) - T^{n-1}(x), a\right\|, \left\|z - T(z),\ a\right\|, \left\|T^{n-1}(x) - T^n(x), a\right\|,\ \frac{\left[\left\|z - T^n(x), a\right\| + \left\|T^{n-1}(x) - T^n(x), a\right\| + \left\|T^n(x) - T(z), a\right\|\right]}{2}\right\}$$

$$= \alpha\left\{\left\|z - T^n(x), a\right\| + \left\|T^n(x) - T^{n-1}(x), a\right\|\right\} \quad [As\ T(z) = z]$$

$$\Rightarrow (1 - \alpha)\ \|z - T^n(x),\ a\| \le \alpha\ \|T^n(x) - T^{n-1}(x),\ a\|$$

$$\Rightarrow \|z - T^n(x),\ a\| \le \frac{\alpha}{1-\alpha}\ \|T^n(x) - T^{n-1}(x),\ a\|$$

Which tends to 0 as $n \to \infty$, since T is asymptotically regular in X. Thus

$$\lim_{n\to\infty} T^n(x) = z$$

3.3. Theorem. Let (X, d) be a 2-Banach space and $\{T_j\}$ be a sequence of mapping of X into itself satisfying

$$\|T_j(x) - T_j(y), a\| \le \alpha \max\left\{\|x-y, a\|, \|x - T_j(x), a\|, \|y - T_j(y), a\|, \frac{[\|x - T_j(y), a\| + \|y - T_j(x), a\|]}{2}\right\} \quad (3.3.1)$$

for every $x, y\ a \in X$, and $\alpha \in (0, 1)$. Suppose $T^n(x) = \lim_{j\to\infty} T_j^n(x)$ for all $x \in X$. If T is asymptotically regular at some point $x \in X$, then T has a unique fixed point in X.

Proof : Let $T^n(x_0) = \lim_{j\to\infty} T_j^n(x)$ for $x_0 \in X$. Then for positive integers m, n

$$\|T_j^m(x_0) - T_j^n(x_0), a\| = \|T_j(T_j^{m-1}(x_0)) - T_j(T_j^{n-1}(x_0)), a\|$$

$$\le \alpha \max\left\{\|T_j^{m-1}(x_0) - T_j^{n-1}(x_0), a\|, \|T_j^{m-1}(x_0) - T_j^m(x_0), a\|, \|T_j^{n-1}(x_0) - T_j^n(x_0), a\|, \frac{[\|T_j^{m-1}(x_0) - T_j^n(x_0), a\| + \|T_j^{n-1}(x_0) - T_j^m(x_0), a\|]}{2}\right\}$$

$$\le \alpha \max\left\{\|T_j^{m-1}(x_0) - T_j^m(x_0), a\| + \|T_j^m(x_0) - T_j^n(x_0), a\| + \|T_j^n(x_0) - T_j^{n-1}(x_0), a\|, \|T_j^{m-1}(x_0) - T_j^m(x_0), a\|, \|T_j^{n-1}(x_0) - T_j^n(x_0), a\|,\right.$$

$$\left.\frac{\left[\left\|T_j^{m-1}(x_0)-T_j^m(x_0),a\right\|+\left\|T_j^m(x_0)-T_j^n(x_0),a\right\|+\left\|T_j^{n-1}(x_0)-T_j^n(x_0),a\right\|+\left\|T_j^n(x_0)-T_j^m(x_0),a\right\|\right]}{2}\right\}$$

$$\leq \alpha\left\{\left\|T_j^{m-1}(x_0)-T_j^m(x_0),a\right\|+\left\|T_j^m(x_0)-T_j^n(x_0),a\right\|+\left\|T_j^n(x_0)-T_j^{n-1}(x_0),a\right\|\right\}$$

$$\Rightarrow (1-\alpha)\left\|T_j^m(x_0)-T_j^n(x_0),a\right\|\leq \alpha\left\{\left\|T_j^{m-1}(x_0)-T_j^m(x_0),a\right\|+\left\|T_j^n(x_0)-T_j^{n-1}(x_0),a\right\|\right\}$$

$$\Rightarrow \left\|T_j^m(x_0)-T_j^n(x_0),a\right\|\leq \frac{\alpha}{1-\alpha}\left[\left\|T_j^{m-1}(x_0)-T_j^m(x_0),a\right\|+\left\|T_j^n(x_0)-T_j^{n-1}(x_0),a\right\|\right]$$

where $\alpha < 1$.

Letting $j \to \infty$ we get

$$\left\|T^m(x_0)-T^n(x_0),a\right\|\leq\left(\frac{\alpha}{1-\alpha}\right)\left[\left\|T^{m-1}(x_0)-T^m(x_0),a\right\|+\left\|T^n(x_0)-T^{n-1}(x_0),a\right\|\right]$$

where $\alpha < 1$.

Let T be asymptotically regular at some point $x_0 \in X$. Then right hand side of the inequality tends to 0 as $m, n \to \infty$ and hence $\{T^n(x_0)\}$ is a Cauchy sequence. Then by completeness of X,

$$\lim_{n\to\infty} T^n(x_0)=u\in X$$

Then

$$\left\|u-T(u),a\right\|\leq\left\|u-T^n(x_0),a\right\|+\left\|T^n(x_0)-T^{n+1}(x_0),a\right\|+\left\|T^{n+1}(x_0)-T(u),a\right\| \qquad (3.3.2)$$

Now

$$\left\|T^{n+1}(x_0)-T(u),a\right\|\leq\alpha\max\left\{\left\|T^n(x_0)-u,a\right\|,\left\|T^n(x_0)-T^{n+1}(x_0),a\right\|,\left\|u-T(u),a\right\|,\right.$$

$$\left.\frac{\left[\left\|T^n(x_0)-T(u),a\right\|+\left\|u-T^{n+1}(x_0),a\right\|\right]}{2}\right\} \quad (3.3.3)$$

Then from (3.3.2) and (3.3.3) we get

$$\left\|u-T(u),a\right\| \le \left\|u-T^n(x_0),a\right\|+\left\|T^n(x_0)-T^{n+1}(x_0),a\right\|$$

$$+\alpha \max\left\{\left\|T^n(x_0)-u,a\right\|,\left\|T^n(x_0)-T^{n+1}(x_0),a\right\|,\left\|u-T(u),a\right\|,\right.$$

$$\left.\frac{\left[\left\|T^n(x_0)-T(u),a\right\|+\left\|u-T^{n+1}(x_0),a\right\|\right]}{2}\right\} \quad (3.3.4)$$

Taking limit of (3.3.4) as $n \to \infty$ we get

$$\|u - T(u),\ a\| \le \alpha\ \|u - T(u),\ a\|$$

and thus we obtain $u = T(u)$.

The proof of uniqueness of u is similar to that proved in Theorem (3.1)

This completes the proof of the theorem 3.3.

Some Related examples :-Following examples show that our theorems 3.1, 3.2. and 3.3 respectively.

Example 3.1. Let $X = R^2$ and consider the following 2-norm on X as $\|x, y\| = |x_1y_2 - x_2y_1|$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then $(X, \|.,.\|)$ is a 2-Banach space. Define self map T on X as follows $Tx = \frac{x}{2}$ for all x. Clearly T is asymptotically regular for all $x \in X$. If we take $\alpha = \frac{1}{2}$ the contractive condition (3.1.1) of theorem 3.1 holds trivially good and 0 is the unique fixed point of the map T.

Example 3.2. Let $X = Q^2$, the field of rational number and consider the following 2-norm on X as $\|x, y\| = |x_1 y_2 - x_2 y_1|$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then $(X, \|.,.\|)$ is a 2-normed space. Define self map T on X as follows $Tx = \frac{x}{2}$ for all x. Clearly T is asymptotically regular for all $x \in X$. Let $\left\{\frac{x}{2^n}\right\}_{n \in N}$ be the sequence of iterates has subsequence $\left\{\frac{x}{2^{nk}}\right\}_{nk \in I^0}$ where I^0 is a set of positive odd integer which converges to 0. If we take $\alpha = \frac{1}{2}$ the contractive condition (3.1.1) holds good and 0 is the unique fixed point of T and sequence $\left\{\frac{x}{2^n}\right\}_{n \in N}$ also converges to 0.

Example 3.3. Let $(X, \|.,.\|)$ be 2-Banach space (defined in example 3.1). Define self map on X as follows $Tx = \frac{x}{2}$ for all x. Clearly T is asymptotically regular for all $x \in X$. Let $\{T_j\}_{j \in N}$ be the sequence of maping where $T_j = \frac{x}{2+\frac{1}{j}}$ then it is obvious that

$$T^n(x) = \lim_{j \to \infty} T_j^n(x) \quad for\ all\ x \in X.$$

If we take $\alpha = \frac{1}{2}$ the contractive condition (3.3.1) of theorem 3.3 holds good and 0 is te unique fixed point of the map T.

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ON HYPERSURFACE OF A FINSLER SPACE WITH AN EXPONENTIAL (α, β) METRIC

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ABSTRACT : The purpose of the present paper is to find certain geometrical properties of the hypersurface of the Finsler space with an exponential (α, β) Metric $\alpha e^{\beta/\alpha}+\frac{\beta^2}{\alpha}$.

Key words : Finsler space, hypersurface, (α, β)-metric..
2000 Mathematics Subject Classification: **53B40.**

1. INTRODUCTION

Let $F^n = (M^n, L)$ be an n-dimensioal finsler space equipped with the fundamental function $L(x, y)$. The normalized supporting element, the metric tensor, the angular metric tensor and Cartan tensor are defined by

$l_i=\dot{\partial}_i L,\ \ g_{ij}=\frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2,\ \ h_{ij}=L\dot{\partial}_i\dot{\partial}_j L$ and $C_{ijk}=\frac{1}{2}\dot{\partial}_k g_{ij}$ respectively, where $\dot{\partial}_i=\frac{\partial}{\partial y^i}$. The Cartan connection in F^n is given as $CT=\left(F^i_{jk}, G^i_j, G^i_{jk}\right)$. The h- and v-covariant derivatives of a covariant vector $X_i(x, y)$ with respect to the Cartan connection are respectively given by

(1.1) $$X_{i/j}=\partial_j X_i-\left(\dot{\partial}_h X_i\right)G^h_j-F^r_{ij}X_r,$$

(1.2) and $$X_{i//j}=\dot{\partial}_j X_i-C^r_{ij}X_r,$$

where $\partial_j=\frac{\partial}{\partial x^j}$.

A Finsler metric $L(x, y)$ is said to be (α, β)-metric if L is positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M^n. Shibata [6] discussed the theory of (α, β)-metric.

M. Matsumoto [4] presented the systematic theory of Finslerian hypersurface. M. K. Gupta and P. N. Pandey [1, 2, 3] obtained certain geometrical properties of the hyper-surfaces of some special Finsler spaces. Lee, Park and Lee [7] discussed the hypersurface of a Finsler spae with metric $\alpha+\frac{\beta^2}{\alpha}$.

In this paper, we consider an n-dimensional Finsler space $F^n = (M^n, L)$ with an exponential (α, β)-metric $\alpha e^{\beta/\alpha}+\frac{\beta^2}{\alpha}$, and find certain geometrical properties of the hyper-surface of the Finsler space with above metric.

The terminologies and notations are referred to Matsumoto [5].

2. PRELIMINARIES

Let M^n be an n-dimensonal smooth manifold and $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with metric $\alpha e^{\beta/\alpha}+\frac{\beta^2}{\alpha}$. The derivative of above exponential metric with respect to α and β are given by

$$L_\alpha = e^{\beta/\alpha}\left(1-\frac{\beta}{\alpha}\right)-\frac{\beta^2}{\alpha^2},\quad L_\beta = e^{\beta/\alpha}+\frac{2\beta}{\alpha},$$

(2.1) $$L_{\alpha\alpha} = \frac{\beta^2}{\alpha^3}\left(e^{\beta/\alpha}+2\right),\quad L_{\beta\beta} = \frac{1}{\alpha}\left(e^{\beta/\alpha}+2\right),$$

$$L_{\alpha\beta} = -\frac{\beta}{\alpha^2}\left(e^{\beta/\alpha}+2\right)$$

where

$$L_\alpha = \frac{\partial L}{\partial\alpha},\quad L_\beta = \frac{\partial L}{\partial\beta},\quad L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial\alpha},\quad L_{\beta\beta} = \frac{\partial L_\beta}{\partial\beta},\quad L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial\beta}.$$

The normalized element of support is given by

(2.2) $$l_i = \alpha^{-1}L_\alpha y_i + L_\beta b_i,$$

where $y_i = a_{ij}y^j$. The angular metric tensor is given by

(2.3) $$h_{ij} = pa_{ij} + q_0 b_i b_j + q_1(b_i y_j + b_j y_i) + q_2 y_i y_j,$$

where

(2.4) $$p = L L_\alpha \alpha^{-1} = \frac{1}{\alpha}\left[(\alpha-\beta)e^{2\beta/\alpha} - \frac{\beta^3}{\alpha^2}e^{\beta/\alpha} - \frac{\beta^4}{\alpha^3}\right],$$

$$q_0 = L L_{\beta\beta} = \left(e^{\beta/\alpha} + \frac{\beta^2}{\alpha^2}\right)\left(e^{\beta/\alpha} + 2\right),$$

$$q_1 = L L_{\alpha\beta}\alpha^{-1} = -\frac{\beta}{\alpha^2}\left(e^{\beta/\alpha} + \frac{\beta^2}{\alpha^2}\right)\left(e^{\beta/\alpha} + 2\right),$$

$$q_2 = L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha \alpha^{-1}) = \frac{1}{a^2}\left(e^{\beta/\alpha} + \frac{\beta^2}{\alpha^2}\right)\left[e^{\beta/\alpha}\left(\frac{\beta^2}{\alpha^2} + \frac{\beta}{\alpha} - 1\right) + \frac{3\beta^2}{\alpha^2}\right].$$

The fundamental metric tensor is given by

(2.5) $$g_{ij} = pa_{ij} + p_0 b_i b_j + p_1(b_i y_j + b_j y_i) + p_2 y_i y_j,$$

where

(2.6) $$p_0 = q_0 + L_\beta^2 = 2e^{2\beta/\alpha} + e^{\beta/\alpha}\left(\frac{\beta^2}{\alpha^2} + \frac{4\beta}{\alpha} + 2\right) + 6\frac{\beta^2}{\alpha^2},$$

$$p_1 = q_1 + L^{-1}pL_\beta = \frac{1}{\alpha}\left[e^{2\beta/\alpha}\left(1 - \frac{2\beta}{\alpha}\right) - e^{\beta/\alpha}\left(\frac{\beta^3}{\alpha^3} + \frac{3\beta^2}{\alpha^2}\right) - \frac{4\beta^3}{\alpha^3}\right],$$

$$p_2 = q_2 + p^2L^{-2} = \frac{1}{\alpha^2}\left[e^{2\beta/\alpha}\left(\frac{2\beta^2}{\alpha^2} - \frac{\beta}{\alpha}\right) + e^{\beta/\alpha}\left(\frac{\beta^4}{\alpha^4} + \frac{3\beta^3}{\alpha^3}\right) + \frac{4\beta^4}{\alpha^4}\right].$$

Moreover, the reciprocal tensor g^{ij} of g_{ij} is given by

(2.7) $$g^{ij} = p^{-1}a^{ij} - s_0 b^i b^j - s_1(b^i y^j + b^j y^i) - s_2 y^i y^j,$$

where

(2.8) $$b^i = a^{ij}b_j,$$

$$s_0 = \left\{pp_0 + \left(p_0 p_2 - p_1^2\right)\alpha^2\right\}/\zeta p,$$

$$s_1 = \left\{pp_1 - \left(p_0 p_2 - p_1^2\right)\beta\right\}/\zeta p,$$

$$s_2 = \left\{pp_2 + \left(p_0 p_2 - p_1^2\right)b^2\right\}/\zeta p, \quad b^2 = a_{ij}b^i b^j,$$

$$\zeta = p\left(p + p_0 b^2 + p_1\beta\right) + \left(p_0 p_2 - p_1^2\right)\left(\alpha^2 b^2 - \beta^2\right).$$

The Cartan tensor C_{ijk} is given by

(2.9) $$2pC_{ijk} = p_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k,$$

where

(2.10) $$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_1 q_0, \quad m_i = b_i - \alpha^{-2}\beta y_i.$$

Let $\left\{{i \atop jk}\right\}$ be the components of Christoffel symbols of the associated Riemannian space R^n *and* ∇_k be the covariant differentiation with respect to x^k relative to this Christoffel symbols. We shall use the following tensors

(2.11) $$2E_{ij} = b_{ij} + b_{ji}, \quad 2F_{ij} = b_{ij} - b_{ji},$$

where $b_{ij} = \nabla_j b_i$.

The difference tensor $D^i_{jk} = F^i_{jk} - \left\{{i \atop jk}\right\}$ of the Finsler space F^n is given by

(2.12) $$D^i_{jk} = B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j}$$

$$-b_{0m}g^{im}B_{jk}-C^{i}_{jm}A^{m}_{k}-C^{i}_{km}A^{m}_{j}+C_{jkm}A^{m}_{s}g^{is}$$

$$+\lambda^{s}\left(C^{i}_{jm}C^{m}_{sk}+C^{i}_{km}C^{m}_{sj}-C^{m}_{jk}C^{i}_{ms}\right),$$

where

(2.13) $$B_k = p_0 b_k + p_1 y_k,\quad B^i = g^{ij}B_j,\quad F^k_i = g^{kj}F_{ji},$$

$$B_{ij}=\left\{p_1\left(a_{ij}-\alpha^{-2}y_i y_j\right)+\frac{\partial p_0}{\partial\beta}m_i m_j\right\}/2,$$

$$A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k,$$

$$B^k_i = g^{kj}B_{ji},\quad \lambda^m = B^m E_{00} + 2B_0 F^m_0,$$

The suffix '0' denotes the transvection by the supporting element y^i except for the quantities p_0, q_0 and s_0.

3. INDUCED CARTAN CONNECTION

A hypersurface M^{n-1} of the underlying manifold M^n may be represented parametrically by the equations $x^i = x^i(u^\alpha)$, where u^α are the Gaussian coordinates on M^{n-1} (Latin indices run from 1 to n, while Greek indices take values from 1 to n-1). We assume that the matrix of projection factors $B^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$ is of rank n–1. If the supporting element y^i at a point $u = (u^\alpha)$ of M^{n-1} is assumed to be tangent to M^{n-1}, we may then write $y^i = B^i_\alpha(u)v^\alpha$ so that $v = (v^\alpha)$ is thought of as the supporting element of M^{n-1} at the point u^α. Since the function $\underline{L}(u, v) = L(x(u), y(u, v))$ gives rise to a Finsler metric on M^{n-1}, we get an (n–1)-dimensional Finsler space $F^{n-1} = (M^{n-1}, \underline{L}(u, v))$, The metric tensor $g_{\alpha,\beta}$ and the Cartan tensor $C_{\alpha\beta\gamma}$ are given by

(3.1) $$g_{\alpha\beta} = g_{ij}B^i_\alpha B^j_\beta,\qquad C_{\alpha\beta\gamma} = C_{ijk}B^i_\alpha B^j_\beta B^k_\gamma.$$

At each point u^α of F^{n-1}, a unit normal vector $N^i(u, v)$ is defined by

(3.2) $$g_{ij}\, B^i_\alpha\, N^j = 0, \quad g_{ij}\, N^i N^j = 1.$$

For the angular metric tensor h_{ij}, we have

(3.3) $$h_{\alpha\beta} = h_{ij}\, B^i_\alpha\, B^j_\beta, \quad h_{ij}\, B^i_\alpha\, N^j = 0, \quad h_{ij} N^i N^j = 1.$$

The inverse projection factors $B^\alpha_i(u, v)$ of B^i_α are defined as

(3.4) $$B^\alpha_i = g^{\alpha\beta}\, g_{ij}\, B^j_\beta,$$

where $g^{\alpha\beta}$ is the inverse of the metric tensor $g_{\alpha\beta}$ of F^{n-1}.

From (3.2) and (3.4), it follows that

(3.5) $$B^i_\alpha\, B^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha\, N_i = 0, \quad N^i B^\alpha_i = 0, \quad N^i N_i = 1,$$

and further

(3.6) $$B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j$$

For the induced Cartan connection $ICT = \left(F^\alpha_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma}\right)$ on F^{n-1}, the second fundamental h-tensor $H_{\alpha\beta}$ and the normal curvature vector H_α are given by

(3.7) $$H_{\alpha\beta} = N_i\left(B^i_{\alpha\beta} + F^i_{jk}\, B^j_\alpha\, B^k_\beta\right) + M_\alpha H_\beta,$$

and

(3.8) $$H_\alpha = N_i\left(B^i_{0\alpha} + G^i_j\, B^j_\alpha\right),$$

where $M_\alpha = C_{ijk}\, B^i_\alpha\, N^j\, N^k$, $B^i_{\alpha\beta} = \partial^2 x^i / \partial u^\alpha\, \partial u^\beta$ and $B^i_{0\alpha} = B^i_{\beta\alpha}\, v^\beta$. It is clear that $H_{\alpha\beta}$ is not symmetric and

(3.9) $$H_{\alpha\beta} - H_{\beta\alpha} = M_\alpha H_\beta - M_\beta H_\alpha.$$

The equations (3.7) and (3.8) yield

(3.10) $$H_{0\alpha} = H_{\beta\alpha}v^{\beta} = H_{\alpha}, \quad H_{\alpha 0} = H_{\alpha\beta}v^{\beta} = H_{\alpha} + M_{\alpha} H_0.$$

The second fundamental v-tensor $M_{\alpha\beta}$ is defined as:

(3.11) $$M_{\alpha\beta} = C_{ijk}\, B^i_{\alpha}\, B^j_{\beta}\, N^k.$$

The relative h- and v-covariant derivatives of B^i_{α} and N^i are given by

(3.12) $$B^i_{\alpha|\beta} = H_{\alpha\beta}N^i, \qquad B^i_{\alpha}|_{\beta} = M_{\alpha\beta}N^i,$$

$$N^i_{|\beta} = -H_{\alpha\beta}\, B^{\alpha}_j\, g^{ij}, \qquad N^i|_{\beta} = -M_{\alpha\beta}\, B^{\alpha}_j\, g^{ij}.$$

Let $X_i(x, y)$ be a vector field of F^n. The relative h- and v-covariant derivatives of X_i are given by

(3.13) $$X_{i|\beta} = X_{i|j}\, B^j_{\beta} + X_i|_j\, N^j H_{\beta}, \quad X_{i|j} = X_{i|j}\, B^j_{\beta}.$$

Matsumoto [4] defined different kinds of hyperplanes and obtained their characteristic conditions, which are given in the following lemmas:

Lemma 3.1. *A hypersurface F^{n-1} is a hyperplane of the first kind if and only if $H_{\alpha} = 0$ or equivalently $H_0 = 0$.*

Lemma 3.2. *A hypersurface F^{n-1} is a hyperplane of the second kind if and only if $H_{\alpha\beta} = 0$.*

Lemma 3.3. *A hypersurface F^{n-1} is a hyperplane of the third kind if and only if $H_{\alpha\beta} = 0 = M_{\alpha\beta}$.*

4. HYPERSURFACE $F^{n-1}(c)$ OF THE FINSLER SPACE WITH THE EXPONENTIAL METRIC

Let us consider the metric $\alpha e^{\beta/\alpha} + \frac{\beta^2}{\alpha}$ with the gradient $b_i(x) = \partial_i b$ for a scalar function $b(x)$ and a hypersurface $F^{n-1}(c)$ given by the equation $b(x) = c$ (constant). From parametric equation

$x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get $\partial_\alpha b(x(u)) = 0 = b_i B^i_\alpha$, so that $b_i(x)$ are regarded as covariant components of a normal vector field of $F^{n-1}(c)$. Therefore along $F^{n-1}(c)$, we have

(4.1) $$b_i B^i_\alpha = 0, \quad b_i y^i = 0.$$

Therefore the induced metric $\underline{L}$ (u, v) of $F^{n-1}(c)$ is given by

(4.2) $$\underline{L}(u,\ v) = \sqrt{\alpha_{\alpha\beta} v^\alpha v^\beta}, \quad a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta,$$

which is a Riemannian metric.

At a point of $F^{n-1}(c)$, from (2.5), (2.7) and (2.9), we have

(4.3) $p = 1$, $q_0 = 3$, $q_1 = 0$, $q_2 = -1/\alpha^2$, $p_0 = 4$, $p_1 = 1/\alpha$, $p_2 = 0$,

$\zeta = 1 + 3b^2$, $s_0 = 3/(1 + 3b^2)$, $s_1 = 1/\alpha\ (1 + 3b^2)$, $s_2 = -b^2/\alpha^2(1 + 3b^2)$.

Therefore (2.8) gives

(4.4) $$g^{ij} = a^{ij} - \frac{3}{\left(1+3b^2\right)} b^i b^j - \frac{1}{\alpha\left(1+3b^2\right)}\left(b^i y^j + b^j y^i\right) + \frac{b^2}{\alpha^2\left(1+3b^2\right)} y^i y^j,$$

using (4.1) we get

$$g^{ij} b_i b_j = \frac{b^2}{1+3b^2}$$

which gives

(4.5) $$b_i(x(u)) = \sqrt{\frac{b^2}{1+3b^2}}\, N_i,$$

where b is the length of the vector b^i. Using (4.4) and (4.5) we get

(4.6) $$b^i = a^{ij} b_j = \sqrt{b^2\left(1+3b^2\right)}\, N^i + b^2 \alpha^{-1} y^i.$$

Theorem 4.1. *Let F^n be a Finsler space equipped with metric* $\alpha e^{\beta/d} + \frac{\beta^2}{\alpha}$ *with a gradient* $b_i(x) = \partial_i b(x)$ *and let* $F^{n-1}(c)$ *be a hypersurface of* F^n*, which is given by* $b(x) = c$ *(constant). Then the induced metric on* $F^{n-1}(c)$ *is Riemannian metric given by (4.2) and the scalar function* $b(x)$ *is given by (4.5) and (4.6).*

Along $F^{n-1}(c)$, the angular metric tensor and the metric tensor of F^n are given by

(4.7) $$h_{ij} = a_{ij} + 3b_ib_j - \frac{y_iy_j}{\alpha^2},$$

(4.8) $$g_{ij} = a_{ij} + 4b_ib_j + \frac{1}{\alpha}\left(b_iy_j + b_jy_i\right).$$

If $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian metric $a_{ij}(x)$, then using (4.1), (4.7) and (3.3), we get

$$h_{\alpha\beta} = h^{(\alpha)}_{\alpha\beta}$$

From (2.7), we get

$$\frac{\partial p_0}{\partial\beta} = 2e^{2\beta/\alpha}\frac{2}{\alpha} + e^{\beta/\alpha}\left(\frac{2\beta}{\alpha^2} + \frac{4}{\alpha}\right) + \frac{1}{\alpha}e^{\beta/\alpha}\left(2 + \frac{\beta^2}{\alpha^2} + \frac{4\beta}{\alpha}\right) + \frac{12\beta}{\alpha^2}.$$

Thus along $F^{n-1}(c)$, $\partial p_0/\partial\beta = 10/\alpha$ and therefore (2.11) gives $\gamma_1 = 1/\alpha$, $m_i = b_i$. Then the Cartan tensor becomes

(4.9) $$C_{ijk} = \frac{1}{2\alpha}\left(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j\right) + \frac{1}{2\alpha}b_ib_jb_k,$$

and therefore using (3.3), (3.11) and (4.1), we get

(4.10) $$M_{\alpha\beta} = \frac{1}{2\alpha}\sqrt{\frac{b^2}{1+3b^2}}h_{\alpha\beta} \text{ and } M_\alpha = 0,$$

and hence from (3.9) it follows that $H_{\alpha\beta}$ is symmetric. Thus we have:

Theorem 4.2. *The second fundamental v-tensor of Finsler hypersurface $F^{n-1}(c)$ of Finsler space equipped with metric $\alpha e^{\beta/\alpha}+\frac{\beta^2}{\alpha}$, is given by* (4.10) *and the second fundamental h-tensor is symmetric.*

Taking h-covarient deivative of (4.1) with respect to the induced connection, we get

$$b_{i|\beta}B^i_\alpha + b_i B^i_{\alpha|\beta} = 0. \tag{4.11}$$

Applying (3.13) for the vector b_i, we get

$$b_{i|\beta} = b_{i|j}\, B^j_\beta + b_i|_j\, N^j\, H_\beta.$$

Using this and $B^i_{\alpha|\beta} = H_{\alpha\beta}\, N^i$, equation (4.11) becomes

$$b_{i|j}\, B^i_\alpha\, B^j_\beta + b_i|_j\, B^i_\alpha\, N^j\, H_3 + b_i\, H_{\alpha\beta} N^i = 0. \tag{4.12}$$

Since $b_i|_j = -b_h\, C^h_{ij}$, using (4.5) and (4.10), we get

$$b_i|_j\, B^i_\alpha\, N^j = -\sqrt{\frac{b^2}{1+3b^2}}M_\alpha = 0.$$

Thus (4.12) gives

$$\sqrt{\frac{b^2}{1+3b^2}}H_{\alpha\beta} + b_{i|j}\, B^i_\alpha\, B^j_\beta = 0. \tag{4.13}$$

Since $H_{\alpha\beta}$ is symmetric, it is clear that $b_{i|j}$ *is symmetric.* Further contracting (4.13) with v^β and then with v^α, we get

$$\sqrt{\frac{b^2}{1+3b^2}}H_\alpha + b_{i|j}\, B^i_\alpha y^j = 0,$$

$$\sqrt{\frac{b^2}{1+3b^2}}H_o + b_{i|j}\, y^i y^j = 0. \tag{4.14}$$

In view of Lemma 3.1, the hypersurface $F^{n-1}(c)$ is a hyperplane of the first kind if and only if $b_{i|j} y^i y^j = 0$. Here $b_{i|j}$ being the covariant derivative with respect to the Cartan connection of F^n may depends on y^i.

Since b_i is a gradient vector, from (2.12), we have $E_{ij} = b_{ij}$, $F_{ij} = 0$. Thus (2.13) reduces to

$$D^i_{jk} = B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s \left(C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}\right), \tag{4.15}$$

In view of (4.3) and (4.4), the relations in (2.14) become to

$$B_i = 4b_i + \frac{1}{\alpha} y_i, \qquad B^i = \frac{3}{1+3b^2} b^i + \frac{1}{\alpha\left(1+3b^2\right)} y^i, \tag{4.16}$$

$$B_{ij} = \frac{1}{2\alpha} a_{ij} - \frac{1}{2\alpha^3} y_i y_j + \frac{5}{\alpha} b_i b_j, \qquad B^i_j = g^{ki} B_{kj},$$

$$A^m_k = B^m_k b_{00} + B^m b_{k0}. \qquad \lambda^m = B^m b_{00}.$$

By virtue of (4.1) we have $B_{i0} = 0$, $B^i_o = 0$ which leads $A^m_0 = B^m b_{00}$. Therefore we have

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - B^m C^i_{jm} b_{00},$$

$$D^i_{00} = B^i b_{00} = \left[\frac{2b^i}{1+2b^2} + \frac{2y^i}{\alpha\left(1+2b^2\right)}\right] b_{00}.$$

Using the relation (4.1), we get

$$b_i D^i_{j0} = \frac{3b^2}{1+3b^2} b_{j0} + \frac{1+10b^2}{2\alpha\left(1+3b^2\right)} b_{00} b_j - \frac{3}{1+3b^2} b^m b_i C^i_{jm} b_{00}, \tag{4.17}$$

$$b_i D^i_{00} = \frac{3b^2}{1+3b^2} b_{00}. \tag{4.18}$$

$b_{i|j}$ is the covariant derivative of b_i with respect to x^j relative to the Cartan connection of F^n and $b_{ij} = \nabla_j b_i$ is the covariant derivative of b_i with respect to x^j relative to the Riemannian connection.

$$b_{i|j} - b_{ij} = \left(\partial_j b_i - F_{ij}^r b_r\right) - \left(\partial_j b_i - \begin{Bmatrix} r \\ ij \end{Bmatrix} b_r\right)$$

$$= -\left(F_{ij}^r - \begin{Bmatrix} r \\ ij \end{Bmatrix}\right) b_r = -D_{ij}^r b_r, \text{ i.e.}$$

(4.19) $$b_{i|j} = b_{ij} - D_{ij}^r br .$$

Using (4.18), we get

$$b_{i|j} y^i y^j = b_{00} - D_{00}^r b_r = \frac{1}{1+3b^2} b_{00}.$$

Consequently, (4.14) may be written as

(4.20) $$\sqrt{\frac{b^2}{1+3b^2}} H_0 + \frac{1}{1+3b^2} b_{00} = 0.$$

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$, where b_{ij} does not depend upon y^i. Since y^i is to satisfy (4.1), the condition is written as $b_{ij} y^i y^j = (b_i y^i)(c_j y^j)$ for some $c_j(x)$, so that we have

(4.21) $$2b_{ij} = b_i c_j + b_j c_i .$$

Using (4.1), it follows that

$$b_{00} = 0, \quad b_{ij} B_\alpha^i B_\beta^j = 0, \quad b_{ij} B_\alpha^i y^j = 0.$$

Again (4.16) and (4.21) gives

$$b_{i0} b^i = \frac{c_0 b^2}{2}, \quad \lambda^m = 0, \quad A_j^i B_\beta^j = 0 \text{ and } B_{ij} B_\alpha^i B_\beta^j = \frac{1}{2\alpha} h_{\alpha\beta}.$$

Using the equations (3.11), (4.4), (4.6), (4.10) and (4.15), we get

$$b_r\, D^r_{ij}\, B^i_\alpha\, B^j_\beta = -\frac{c_0 b^2}{4\alpha\left(1+3b^2\right)^2} h_{\alpha\beta}.$$

Therefore in view of (4.19), the equation (4.13) reduces to

$$\sqrt{\frac{b^2}{1+3b^2}}H_{\alpha\beta} + \frac{c_0 b^2}{4\alpha\left(1+3b^2\right)^2} h_{\alpha\beta} = 0. \tag{4.22}$$

Theorem 4.3. *The necessary and sufficient condition for the hypersurface $F^{n-1}(c)$ of Finsler space equipped with metric $\alpha e^{\beta/\alpha} + \frac{\beta^2}{\alpha}$, to be hyperplane of the first kind is (4.21) and in this case the second fundamental h-tensor of hypersurface $F^{n-1}(c)$, is proportional to its angular metric tensor.*

In view of Lemma 3.2, the hypersurface $F^{n-1}(c)$ is a hyperplane of the second kind if and only if $H_{\alpha\beta} = 0$. Thus from (4.22) we get $c_0 = c_i(x)y^i = 0$. Therefore there exists a function $e(x)$ such that $c_i(x) = e(x)b_i(x)$. Thus (4.21) gives

$$b_{ij} = e\, b_i b_j. \tag{4.23}$$

Theorem 4.4. *The necessary and sufficient condition for the hypersurface $F^{n-1}(c)$ of Finsler space equipped with metric $\alpha e^{\beta/\alpha} + \frac{\beta^2}{\alpha}$, to be hyperplane of the second kind is (4.23).*

In view of equation (4.10) and Lemma 3.3, we have:

Theorem 4.5. *The hypersurface $F^{n-1}(c)$ of Finsler space equipped with metric $\alpha e^{\beta/\alpha} + \frac{\beta^2}{\alpha}$ is not a hyperplane of the third kind.*

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ON THREE-DIMENSIONAL NORMAL ALMOST CONTACT METRIC MANIFOLDS

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ABSTRACT : In this paper we study Ricci solitions and gradient Ricci solitions on a three-dimensional normal almost cfontact metric manifold admitting quarter symmetric non-metric connection. We prove that on a three-dimensional normal almost contact metric manifold with qurter symmetric non-metric connection, Ricci solition with a potential vector field V collinear with the characteristic vector field ξ has constant scalar curvature provided α, β = constant. Also we investigate gradient Ricci solitions on a three-dimensional normal almost contact metric manifold admitting quarter symmetric non-metric connection. Finally, we study a three-dimensional normal almost contact metric manifold admitting quarter symmetric non-metric connection which satisfies $\widetilde{R}\cdot\widetilde{S}=0$, $\widetilde{P}\cdot\widetilde{S}=0$ and $\widetilde{Z}\cdot\widetilde{S}=0$ with Ricci solutions

Key words and phrases : Normal almost contact metric manifold. Ricci gradient soliton, quarter symmetric non-metric connection, constant scalar curvature.
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1. INTRODUCTION

A Ricci solition is a generalization of an Einstein metric. In a Riemannian manifold (M, g), g is called a Ricci solition if [20]

$$£vg + 2S + 2\lambda g = 0, \tag{1.1}$$

where $£$ is the Lie derivative, S is the Ricci tensor, V is a smooth vector field on M (called the potential vector field) and λ is a constant. Metrics satisfying (1.1) are interesting and useful in physics and are often referred as quasi-Einstein (e.g. [7], [8], [16]. Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of the Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan who discusses some aspects of it in [16].

The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive, respectively. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci solition and quation (1.1) assumes the form

$$\nabla\nabla f = S + \lambda g. \tag{1.2}$$

A Ricci soliton on a compact manifold has constant curvature in dimension 2 [20] and also in dimension 3 [21]. For details we refer to Chow and Knopf [9]. We also recall the following significant result of Perelman [25]: a Ricci soliton on a compact manifold is a gradient Ricci soliton.

In [26], Sharma has started to study of Ricci solitons in K-contact manifolds. Also, in a subsequent paper [18] Chosh, Sharma and Cho studied the gradient Ricci soliton of a non-Sassakian (k, μ)-contact manifold. Also in [4] C. Calin and M. Crasmareanu have studied the Ricci solitons in f-Kenmotsu manifolds. Then De et al. [10] have studied the Ricci solitons and gradient Ricci solitons on three-dimensional normal almost contact metric manifolds.

The idea of quarter symmetric linear connection in a differentiable manifold was introduced by Golab [19]. A linear connection $\overline{\nabla}$ on an n-dimensional Riemannian manifold (M, g) is called a quarter symmetric connection if its torsion tensor T of the connection $\overline{\nabla}$

$$T(X, Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y], \tag{1.3}$$

satisfies

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.4}$$

where η is a 1-form and ϕ is a (1, 1)-tensor field. In particular, if $\phi(X) = X$, then the quarter symmetric connection reduces to a semi-smmetric connection [17]. Thus the notion of quarter symmetric connection generalizes the idea of the semi-symmetric connection. If moreover, a quarter symmetric connection $\overline{\nabla}$ satisfies the condition

$$(\overline{\nabla}_X g)(Y, Z) \neq 0, \tag{1.5}$$

for all $X, Y, Z \in \mathfrak{X}(M)$, then $\overline{\nabla}$ is said to be a quarter symmetric non-metric connection, otherwise it is said to be a quarter symmetric metric connection. Several authors have studied

quarter symmetric non-metric connection. Some of them are [5], [15], [12], [13] and [6]. Here we use quarter symmetric non-metric connection $\overline{\nabla}$ such that

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X \tag{1.6}$$

which satisfies equations (1.3) and (1.4).

The paper is organised as follows: After some preliminiaries, in Section 4, we show that in a three-dimensional normal almost contact metric manifold admitting quarter symmetric non-metric connection with the Ricci solition g, if the potential vector field V collinear with the characteristic vector field ξ, then the manifold has constant scalar curvature, provided $\alpha = \beta =$ constant. In Section 5, we show that if a three-dimensional normal almost contact metric manifold admitting quarter symmetric non-metric connecton with the gradient Ricci soliton then the manifold is either α-Sasakian or an Einstein manifold provided $\alpha = \beta =$ constant. In the last section, we investigate some curvature conditions (i.e. $\widetilde{R}\cdot\widetilde{S}=0$, $\widetilde{P}\cdot\widetilde{S}=0$ and $\widetilde{Z}\cdot\widetilde{S}=0$) in a three-dimensional normal almost contact metric manifold admitting quarter symmetric non-metic connection with the Ricci soliton.

2. PRELIMINIARIES

Let M be an almost contact manifold and (ϕ, ξ, η) its almost contact structure. This means, M is an odd-dimensional differentiable manifold and ϕ, ξ, η are tensor field on M of types (1, 1), (1, 0), (0, 1), respectively, such that

$$\phi^2 = -I + \eta\otimes\xi, \ \eta(\xi) = 1.$$

Then also $\phi\xi = 0$, $\eta \circ \phi = 0$. Let $\mathbf{R}$ be real line and t a coordinate on $\mathbf{R}$. Define an almost complex struture J on $M \times \mathbf{R}$ by

$$J\left(X, \frac{fd}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

where the pair $(X, fd/dt)$ denotes a tangent vetor to $M \times \mathbf{R}$, f is a smooth function on $M \times \mathbf{R}$, X and fd/dt being tangent to M and $\mathbf{R}$, respectively.

M and (ϕ, ξ, η) are said to be normal if the structure J is integrable [1], [2]. The neessary and sufficient condition for (ϕ, ξ, η) to be normal is

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ defined by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y],$$

for any $X, Y \in X(M)$, $X(M)$ being the Lie algebra of vector fields on M.

A Riemannian metric g on M satisfying the condition

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any $X, Y \in X(M)$, is said to be compaible with the structure (ϕ, ξ, η). If g is such a metric, then the quadruple (ϕ, ξ, η, g) is called an almost contact metric strucuure on M and M is an almost contact metric manifold. On such a manifold we also have $\eta(X) = g(X, \xi)$ for any $X \in X(M)$ and we can always define the fundamental 2-form Φ by $\Phi(X, Y) = g(X, \Phi Y)$, where $X, Y \in X(M)$.

It is no hard to see that if $\dim M = 3$, then two Riemannian metrics g and g' are compatible with the same almost positive function σ on M.

A normal almost contact metric structure (ϕ, ξ, η, g) satisfying additionally the condition $d\eta = \Phi$ is called Sasakian. Also a normal almost contact metric structure satisfying the condition $d\Phi = 0$ is said to be quasi-Sasakian [3].

For an almost contact metric structure (ϕ, ξ, η, g) on M, we have [23]:

$$(\nabla_X \phi)(Y) = g(\phi\nabla_X\xi, Y)\xi - \eta(Y)\phi\nabla_X\xi, \tag{2.1}$$

$$\nabla_X\xi = \alpha\{X - \eta(X)\xi\} - \beta\phi X, \tag{2.2}$$

$$(\nabla_X\eta)(Y) = \alpha\{g(X, Y) - \eta(X)\eta(Y)\} - \beta g(\phi X, Y), \tag{2.3}$$

where $2\alpha = div\xi$ and $2\beta = tr(\phi\nabla\xi)$, $div\xi$ is the divergence of ξ defined by $div\xi = trace\{x \to \nabla_X\xi\}$ and $tr(\phi\nabla\xi) = trace\{X \to \phi\nabla_X\xi\}$.

(2.4) $$R(X, Y)\xi = \{Y\alpha + (\alpha^2 - \beta^2)\eta(Y)\}\phi^2 X - \{X\alpha + (\alpha^2 - \beta^2)\eta(X)\}\phi^2 Y + \{Y\beta + 2\alpha\beta\eta(Y)\}\phi X - \{X\beta + 2\alpha\beta\eta(X)\}\phi Y,$$

(2.5) $$S(Y, \xi) = -Y\alpha - (\phi Y)\beta - \{\xi\alpha + 2(\alpha^2 - \beta^2)\}\eta(Y),$$

(2.6) $$\xi\beta + 2\alpha\beta = 0,$$

where R denotes the curvature tensor and S is the Ricci tensor.

One the other hand, the curvature tensor in dimension three always satifies [27]

(2.7) $$\widetilde{R}(X, Y, Z, W) = g(X, W)S(Y, Z) - g(X, Z)S(Y, W) + g(Y, Z)S(X, W) - g(Y, W)S(X, Z) - \frac{r}{2}[g(X, W)g(Y, Z) - g(X, Z)g(Y, W)],$$

where $\widetilde{R}(X, Y, Z, W) = g(R(X, Y)Z, W)$ and r is the scalar curvature.

From (2.4), we can derive that

(2.8) $$\widetilde{R}(\xi, Y, Z, \xi) = -(\xi\alpha + \alpha^2 - \beta^2)g(\phi Y, \phi Z) - (\xi\beta + 2\alpha\beta)g(Y, \phi Z).$$

By (2.5), (2.7) and (2.8), we obtain for α, β = constant,

(2.9) $$S(Y, Z) = (\frac{r}{2} + \alpha^2 - \beta^2)g(\phi Y, \phi Z) - 2(\alpha^2 - \beta^2)\eta(Y)\eta(Z),$$

and

(2.10) $$QY = (\frac{r}{2} + \alpha^2 - \beta^2)Y - (\frac{r}{2} - \alpha^2 + \beta^2)\eta(Y)\xi.$$

Applying (2.9) in (2.7), we get

(2.11) $$R(X, Y)Z = [\frac{r}{2} + 2(\alpha^2 - \beta^2)][g(Y, Z)X - g(X, Z)Y] + g(X, Z)[(\frac{r}{2} + 3(\alpha^2 - \beta^2))\eta(Y)\xi] - [(\frac{r}{2} + 3(\alpha^2 - \beta^2)]\eta(Y)\eta(Z)X$$

$$- g(Y, Z)[(\frac{r}{2} + 3(\alpha^2 - \beta^2))\eta(X)\xi]$$

$$+ [(\frac{r}{2} + 3(\alpha^2 - \beta^2)]\eta(X)\eta(Z)Y.$$

It is to be noted that the general formulas can be obtained by straightforward calculation.

Several authors have studied three-dimensional normal almost contact metric manifolds, such as ([11], [14]) and many others.

Also from (2.6) it follows that if α, β = constant, then the manifold is either β-Sasakian, or α-Kenmotsu, or cosymplectic [22].

Proposition 1. *A three-dimensional normal almost contact metric manifold with* α, β = *constant is either* β*-Sasakian, or* α*-Kenmotsu or cosymplectic.*

We note that β-Sasakian manifolds are quasi Sasakian ([3], [24]).

3. QUARTER SYMMETRIC NON-METRIC CONNECTION IN A THREE-DIMENSIONAL NORMAL ALMOST CONTACT METRIC MANIFOLD

In this section, we give equations of a three-dimensional normal almost contact metric manifold admiting quarter symmetric non-metric connections, where $\overline{\nabla}$ is a quarter symmetric non-metric connection on a three-dimensional normal almost contact metric manifold.

Let ∇ be linear connection on a three-dimensional normal almost contact metric manifold. We define a connection

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X. \tag{3.1}$$

Then the above connection satisfies (1.3) and (1.4). For a three-dimensional normal almost contact metric manifold admitting quarter symmetric non-metric connection we have the following:

$$\begin{aligned} \overline{R}(X, Y)Z = R(X, Y)Z &+ \alpha g(Z, X)\phi Y - \beta g(\phi X, Z)\phi Y \\ &+ 2\alpha\eta(Z)g(\phi X, Y)\xi - \beta\eta(Z)\eta(Y)X - \alpha g(Z, Y)\phi X \\ &+ \beta g(Z, \phi Y)\phi X + \beta\eta(X)\eta(Z)Y, \end{aligned} \tag{3.2}$$

(3.3) $$\bar{S}(Y, Z) = S(Y, Z) + \beta g(Y, Z) + \alpha g(\phi Y, Z) - 3\beta\eta(Y)\eta(Z),$$

(3.4) $$\bar{r} = r.$$

4. RICCI SOLITON

Let V be point-wise collinear with ξ i.e. $V = b\xi$, where b is a function on a three-dimensional normal almost contact metric manifold admitting quarter symmetric non-metric connection Then

(4.1) $$(\pounds_V g + 2\bar{S} + 2\lambda g)(X, Y) = 0.$$

or,

(4.2) $$g(\nabla_X V, Y) + \eta(V)g(\phi X, Y) + g(\nabla_Y V, X) + \eta(V)g(\phi Y, X) + 2S(X, Y) + 2\alpha g(\phi X, Y) + 2\beta g(X, Y) - 6\beta\eta(X)\eta(Y) + 2\lambda g(X, Y) = 0,$$

which implies that

(4.3) $$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X, Y) + \{2\lambda + 2\beta\}g(X, Y) + 2\alpha g(\phi X, Y) - 6\beta\eta(X)\eta(Y) = 0,$$

or,

(4.4) $$bg(\nabla_X\xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y\xi, X) + (Yb)\eta(X) + 2S(X, Y) + (2\lambda + 2\beta)g(X, Y) + 2\alpha g(\phi X, Y) - 6\beta\eta(X)\eta(Y) = 0.$$

Using (2.2) in (4.4), we obtain

(4.5) $$(Xb)\eta(Y) + (Yb)\eta(X) + 2b\alpha(g(X, Y) - \eta(X)\eta(Y)) + 2S(X, Y) + (2\lambda + 2\beta)g(X, Y) + 2\alpha g(\phi X, Y) - 6\beta\eta(X)\eta(Y) = 0.$$

In (4.5) replacing Y by ξ and using (2.9), it follows that

(4.6) $$(Xb) + (\xi b)\eta(X) - 4(\alpha^2 - \beta^2)\eta(X) + (2\lambda + 2\beta)\eta(X) - 6\beta\eta(X) = 0$$

Putting $X = \xi$ in (4.6), we get

$$\xi b = 2(\alpha^2 - \beta^2) - \lambda + 2\beta. \tag{4.7}$$

Using (4.7) in (4.6), we obtain

$$db = (2(\alpha^2 - \beta^2) - \lambda + 2\beta)\eta. \tag{4.8}$$

Applying d on (4.8), we get

$$(2(\alpha^2 - \beta^2) - \lambda + 2\beta)d\eta = 0. \tag{4.9}$$

Since $d\eta \neq 0$, we have

$$2(\alpha^2 - \beta^2) - \lambda + 2\beta = 0. \tag{4.10}$$

Using (4.10) in (4.8) yields b is a constant. Therefore from (4.4), it follows

$$\begin{aligned} S(X, Y) &= 3\beta\eta(X)\eta(Y) - \alpha g(\phi X, Y) \\ &\quad - (\lambda + \beta)g(X, Y) - b\alpha\{g(X, Y) - \eta(X)\eta(Y)\}. \end{aligned} \tag{4.11}$$

In (4.11) taking $X = Y = e_i$, we find

$$r = -3\lambda - 2b\alpha.$$

Hence we can state the followings.

Theorem 1. *Let M be a three-dimensional normal almost contact metric manifold admitting quarter symmetric non-metric connection. If the metric g is the Ricci soliton and V is point-wise collinear with ξ, then V is a constant multiple of ξ and the manifold has constant scalar curvature provided that α, β = constant.*

Theorem 2. *Let M be a three-dimensional normal almost contact metric manifold admitting quarter symmetric non-metric connection. If the metric g is the Ricci soliton and V is point-wise collinear with ξ, then the Ricci soliton is expanding, steady and shrinking according as $r + 2b\alpha < 0$, $r + 2b\alpha = 0$ and $r + 2b\alpha > 0$, respectively.*

Now let be $V = \xi$. Then the equation (4.1) reduces to

$$g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X) + 2S(X, Y) + (2\lambda + 2\beta)g(X, Y)$$
$$+ 2\alpha g(\phi X, Y) - 6\beta\eta(X)\eta(Y) = 0.$$

Using (2.9) in the above equation and putting $X = Y = \xi$, we obtain

$$\lambda = 2(\alpha^2 - \beta^2) + 2\beta.$$

Thus we have the following:

Theorem 3. *Let M be a three-dimensional normal almost contact metric manifold admitting quarter symmetric non-metric connection. If the metric g is the Ricci soliton and $V = \xi$, then the Ricci solition is expanding, steady and shrinking according as* $(\alpha^2 - \beta^2) + \beta > 0$, $(\alpha^2 - \beta^2) + \beta = 0$ *and* $(\alpha^2 - \beta^2) + \beta < 0$, *respectively.*

5. GRADIENT RICCI SOLITON

If the vector field V is the gradient of a potential function $-f$ on a three-dimensional normal almost contact metric manifold admitting quarter symmetric non-metric connection, then g is called a gradient Ricci soliton and (4.1) assume the form

(5.1) $$\overline{\nabla}\nabla f = \nabla\nabla f = \overline{S} + \lambda g.$$

This reduces to

(5.2) $$\overline{\nabla}_X Df = \overline{Q}X + \lambda X,$$

where D denotes the gradient operator of g. From (5.2), it is clear that

(5.3) $$\overline{R}(X, Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X$$
$$+ (6\beta^2 + 2\alpha^2)g(\phi X, Y)\xi - 4\beta\alpha(\eta(Y)X - \eta(X)Y)$$
$$- (-\lambda + 4\alpha^2 + 3\beta - 5\beta^2)\{\eta(Y)\phi X - \eta(X)\phi Y\}.$$

Differentiating (2.10), we have

(5.4) $$(\nabla_W Q)(X) = \frac{dr(W)}{2}(X - \eta(X)\xi)$$
$$-\left(\frac{r}{2} + 3(\alpha^2 - \beta^2)\right)\left\{(\nabla_W\eta)(X)\xi - \eta(X)\nabla_W\xi\right\}$$

In (5.4) replacing W by ξ yields

$$(5.5)\qquad (\nabla_\xi Q)(X) = \frac{d\tau(\xi)}{2}(X - \eta(X)\xi)$$

$$-\left(\frac{\tau}{2} + 3\left(\alpha^2 - \beta^2\right)\right)\left\{\left(\nabla_\xi \eta\right)(X)\xi - \eta(X)\nabla_\xi \xi\right\}$$

Then we have

$$(5.6)\qquad g(\nabla_\xi Q)Y - (\nabla_Y Q)\xi + (6\beta^2 + 2\alpha^2)g(\phi\xi, Y)\xi - 4\beta\alpha(\eta(Y)\xi - \eta(X)\xi)$$

$$-(-\lambda + 4\alpha^2 + 3\beta - 5\beta^2)\{\eta(Y)\phi\xi - \eta(\xi)\phi Y\}, \xi) = 0.$$

Using (5.6) from (5.3), we obtain

$$(5.7)\qquad g((\overline{R}(\xi, Y)Df, \xi) = 0.$$

From (2.8), we get

$$(5.8)\qquad g(\overline{R}(\xi, Y)Df, \xi) = (\alpha^2 - \beta^2)\{-g(Y, Df) + \eta(Y)\eta(Df)\} + 2\alpha\beta g(\phi Y, Df).$$

Using (5.7) in (5.8) yields

$$(5.9)\qquad (\alpha^2 - \beta^2)\{-g(Y, Df) + \eta(Y)\eta(Df)\} + 2\alpha\beta g(\phi Y, Df) = 0.$$

Replacing Y by ϕY in (5.9), we have

$$-(\alpha^2 - \beta^2)\{-g(Y, Df) + \eta(Y)\eta(Df)\} - 2\alpha\beta g(Y, Df) + 2\alpha\beta g(\xi, Df) = 0$$

Replacing the value of $g(\phi X, Df)$ in (5.9), we obtain

$$(5.10)\qquad Df = (\xi f)\xi, \text{ since } \alpha^2 \neq \beta^2.$$

Using (5.10) in (5.2), we get

$$\overline{S}(X, Y) + \lambda g(X, Y) = g(\overline{\nabla}_Y Df, X) = g(\overline{\nabla}_Y(\xi f)\xi, X)$$

$$(5.11)\qquad = g(\nabla_Y(\xi f)\xi, X) + (\xi f)g(\phi Y, X)$$

$$= g(Y(\xi f)\xi + (\xi f)\nabla_Y\xi, X) + (\xi f)g(\phi Y, X)$$

$$= Y(\xi f)\eta(X) + (\xi f)(1 - \beta)g(\phi Y, X) + \alpha(\xi f)(g(X, Y) - \eta(Y)\eta(X))$$

Putting $X = \xi$ in (5.11) and using (2.9), we have

$$\bar{S}(Y, \xi) + \lambda\eta(Y) = Y(\xi f) = \{\lambda - 2\beta - 2(\alpha^2 - \beta^2)\}\eta(X). \tag{5.12}$$

Interchanging X and Y in (5.11), we obtain

$$\begin{aligned} \bar{S}(X, Y) + \lambda g(X, Y) &= X(\xi f)\eta(Y) + (\xi f)(1 - \beta)g(\phi X, Y) \\ &\quad + \alpha(\xi f)(g(X, Y) - \eta(Y)\eta(X)) \end{aligned} \tag{5.13}$$

Adding (5.11) and (5.13), we get

$$\begin{aligned} \bar{S}(X, Y) + \lambda g(X, Y) &= \alpha(\xi f)g(X, Y) \\ &\quad + \{\lambda - 2\beta - 2(\alpha^2 - \beta^2) - \alpha(\xi f)\}\eta(Y)\eta(X). \end{aligned} \tag{5.14}$$

Then using (5.2), we have

$$\bar{\nabla}_Y Df = \alpha(\xi f)Y + \{\lambda - 2\beta - 2(\alpha^2 - \beta^2) - \alpha(\xi f)\}\eta(Y)\xi. \tag{5.15}$$

Using (5.15), we calculate

$$\begin{aligned} \bar{R}(X, Y) &= \bar{\nabla}_X\bar{\nabla}_Y Df - \bar{\nabla}_Y\bar{\nabla}_X Df - \bar{\nabla}_{[X,Y]}Df \\ &= \alpha X(\xi f)Y - Y(\xi f)X + \alpha(\xi f)\eta(Y)\phi X - \alpha(\xi f)\eta(X)\phi Y \\ &\quad + \{\lambda - 2\beta - 2(\alpha^2 - \beta^2) - \alpha(\xi f)\}(\alpha\eta(Y)X - \alpha\eta(X)Y \\ &\quad + \eta(Y)\phi X - \eta(X)\phi Y - \beta\eta(Y)\phi X + \beta\eta(X)\phi Y - 2\beta g(Y, \phi X)\xi). \end{aligned} \tag{5.16}$$

Taking inner product with ξ in (5.16), we get

$$0 = g(\bar{R}(X, Y)Df, \xi) = 2\beta\{\lambda - 2\beta - 2(\alpha^2 - \beta^2) - \alpha(\xi f)\}g(\phi Y, X).$$

Thus we obtain

$$\beta\{\lambda - 2\beta - 2(\alpha^2 - \beta^2) - \alpha(\xi f)\} = 0.$$

We have the followings:

Case i) $\beta = 0$,

Case ii) $\lambda - 2\beta - 2(\alpha^2 - \beta^2) - \alpha(\xi f) = 0$,

Case iii) $\beta = 0$ and $\lambda - 2\beta - 2(\alpha^2 - \beta^2) - \alpha(\xi f) = 0$.

Now we consider these cases:

Case i) If $\beta = 0$, then the manifold reduces to a α-Kenmotsu manifold.

Case ii) Let $\lambda - 2\beta - 2(\alpha^2 - \beta^2) - \alpha(\xi f) = 0$. If we use this in (5.13) gives

$$Y(\xi f) = \alpha(\xi f)\eta(Y).$$

Substitute this value in (5.14), we obtain

$$\bar{S}(X, Y) + \lambda g(X, Y) = \alpha(\xi f)/(X, Y).$$

Contracting this equation, we get

$$\bar{r} + 3\lambda = 3\alpha(\xi f),$$

which implies that

$$(\xi f) = \frac{\bar{r}}{3\alpha} + \frac{\lambda}{\alpha}.$$

If $\bar{r}$ = constant, then (ξf) =constant = c. Therefore from (5.10), we have

$$Df = (\xi f)\xi = c\xi.$$

Thus we can write from this equation

$$g(Df, X) = c\eta(X),$$

which means that

$$df(X) = c\eta(X).$$

Applying d on the above equation, we get

$$cd\eta = 0.$$

Since $d\eta \neq 0$, we have $c = 0$. Hence we get $Df = 0$. This means that f = constant. Therefore equation (5.13) reduces

$$\bar{S}(X, Y) = (2(\alpha^2 - \beta^2) + 2\beta)g(X, Y),$$

that is, M is an Einstein manifold admitting quarter symmetric non-metric connection.

Case iii) using $\beta = 0$ and $\lambda - 2\beta - 2(\alpha^2 - \beta^2) - \alpha(\xi f) = 0$ in (5.13), we obtain $Y(\xi f) = \alpha(\xi f)\eta(Y)$. Now as in *Case ii)* we conclude that the manifold is an Einstein manifold.

Thus we have the following:

Theorem 4. *If a three-dimensional normal almost contct metric manifold admitting quarter symmetric non-metric connection with the gradient Ricci soliton then the manifold is either* α*-Kenmotsu or an Einstein manifold provided* α, β = *constant.*

6. THREE-DIMENSIONAL NORMAL ALMOST CONTACT METRIC MANIFOLDS ADMITTING QUARTER SYMMETRIC NON-METRIC CONNECTION WITH THE RICCI SOLITION SATISFYING $\tilde{R}\cdot\tilde{S}=0$, $\tilde{P}\cdot\tilde{S}=0$ AND $\tilde{Z}\cdot\tilde{S}=0$.

Let R, P, S and Z be Riemannian curvature tensor, the projective curvature tensor, the Ricci tensor and the concircular curvature tensor with respect to ∇ and $\tilde{R}$, $\tilde{P}$, $\tilde{S}$ and $\tilde{Z}$ be Riemannian curvature tensor, the projective tensor, the Ricci tensor and the concircular curvature tensor with respect to ∇ on a three-dimensional normal almost contact metric manifold M with the Ricci solition and quarter symmetric non-metric connection $\overline{\nabla}$.

Using (2.11) and (4.1) with $V = \xi$, we have the followings:

(6.1) $$\tilde{S}(X, Y) = -(\alpha + \lambda + \beta)g(X, Y) + (3\beta + \alpha)\eta(X)\eta(Y) - \alpha g(\phi X, Y),$$

(6.2) $$\tilde{r} = 3\lambda - 2\alpha,$$

(6.3) $$\tilde{Q}X = -(\alpha + \lambda + \beta)X + (3\beta + \alpha)\eta(X)\xi - \alpha\phi X,$$

(6.4) $$\begin{aligned}\tilde{R}(X,Y)Z = {} & (-2\alpha - 2\lambda - 2\beta - \tfrac{r}{2})\{g(Y, Z)X - g(X, Z)Y\} \\ & + (3\beta + \alpha)\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ & + g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi\} \\ & - \alpha\{g(\phi Y, Z)X - g(\phi X, Z)Y + g(Y, Z)\phi X - g(X, Z)\phi Y\},\end{aligned}$$

(6.5) $$\begin{aligned}\tilde{P}(X,Y)Z = {} & (-2\alpha - 2\lambda - 2\beta - \tfrac{r}{2})\{g(Y, Z)X - g(X, Z)Y\} \\ & + (3\beta + \alpha)\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\end{aligned}$$

$$+ g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi\}$$

$$- \alpha\{g(\phi Y, Z)X - g(\phi X, Z)Y + g(Y, Z)\phi X - g(X, Z)\phi Y\},$$

$$- \frac{1}{n-1}\{(3\beta + \alpha)\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}$$

$$- (\beta + \alpha + \lambda)\{g(Y, Z)X - g(X, Z)Y\}$$

$$- \alpha(g(\phi\phi Y, Z)X - g(\phi X, Z)Y)\}.$$

$$\tilde{Z}(X, Y)W = (-2\alpha - 2\lambda - 2\beta - \tfrac{r}{2} - \tfrac{r}{n(n-1)})\{g(Y, W)X - g(X, W)Y\} \tag{6.6}$$

$$+ (3\beta + \alpha)\{\eta(Y)\eta(W)X - \eta(X)\eta(W)Y$$

$$+ g(Y, W)\eta(X)\xi - g(X, W)\eta(Y)\xi\}$$

$$- \alpha\{g(\phi Y, W)X - g(\phi X, W)Y + g(Y, W)\phi X - g(X, W)\phi Y\}.$$

Using above equations, we investigate the conditions $\tilde{R}\cdot\tilde{S} = 0$, $\tilde{P}\cdot\tilde{S} = 0$ and $\tilde{Z}.\tilde{S} = 0$.

Let be $\tilde{R}(X, Y)\cdot\tilde{S} = 0$. Then

$$(\tilde{R}(X, Y)\tilde{S})(Z, W) = -\tilde{S}(\tilde{R}(X, Y)Z, W) - \tilde{S}(Z, \tilde{R}(X, Y)W) = 0.$$

Now using (6.1) in the above equation, we get

$$0 = (-\alpha - \lambda - \beta)g(\tilde{R}(X, Y)Z, W) + (3\beta + \alpha)\eta(\tilde{R}(X, Y)Z)\eta(W) \tag{6.7}$$

$$- \alpha g(\phi\tilde{R}(X, Y)Z, W) + (-\alpha - \lambda - \beta)g(Z, \tilde{R}(X, Y)W)$$

$$+ (3\beta + \alpha)\eta(Z)\eta(\tilde{R}(X, Y)W) - \alpha g(\phi Z, \tilde{R}(X, Y)W).$$

Using (6.4) in (6.7), we obtain

$$0 = (3\beta + \alpha)(\beta - \alpha - 2\lambda - \tfrac{r}{2})\{g(Y, Z)\eta(X)\eta(W) - g(X, Z)\eta(Y)\eta(W) \tag{6.8}$$

$$+ g(Y, W)\eta(X)\eta(Z) - g(X, W)\eta(Y)\eta(Z)\}$$

$$- 2\alpha\{\alpha + 3\beta\}\{g(\phi Y, Z)\eta(X)\eta(W) - g(\phi X, Z)\eta(Y)\eta(W)\}.$$

Putting $X = Z = e_i$ in (6.8), we find

$$0 = (3\beta + \alpha)(\beta - \alpha - 2\lambda - \tfrac{r}{2})\{g(Y, W) - 3\eta(Y)\eta(W)\}. \tag{6.9}$$

Putting $Y = \xi$ in (6.9), we have

$$0 = (3\beta + \alpha)(\beta - \alpha - 2\lambda - \tfrac{r}{2}),$$

that is, either

$$\alpha = -3\beta,$$

or,

$$\lambda = \frac{\beta - \alpha - \frac{r}{2}}{2}.$$

Thus we give the following:

Theorem 5. *Let M be a three-dimensional normal almost contact metric manifold admitting quarter symmetric non-metric connection which admits the Ricci soliton for $V = \xi$. If $\widetilde{R}(X,Y)\cdot\widetilde{S} = 0$, then either $\lambda = \frac{\beta - \alpha - \frac{r}{2}}{2}$ or, $\alpha = -3\beta$, provided α, β = constant.*

Now we consider the condition $\widetilde{P}(X,Y)\cdot\widetilde{S} = 0$. Then

$$(\widetilde{P}(X,Y)\widetilde{S})(Z,W) = -\widetilde{S}(\widetilde{P}(X,Y)Z,W) - \widetilde{S}(Z,\widetilde{P}(X,Y)W).$$

Using (6.5) and (6.1) in the above equation, we have

$$\begin{aligned} 0 = {} & (3\beta + \alpha)(\beta - \alpha - 2\lambda - \tfrac{r}{2})\{g(Y, Z)\eta(X)\eta(W) - g(X, Z)\eta(Y)\eta(W) \\ & + g(Y, W)\eta(X)\eta(Z) - g(X, W)\eta(Y)\eta(Z)\} \\ & - 2\alpha\{\alpha + 3\beta\}\{g(\phi Y, Z)\eta(X)\eta(W) - g(\phi X, Z)\eta(Y)\eta(W)\} \\ & - \tfrac{1}{n-1}\{2\alpha g(\phi Z, Y)\{(3\beta + \alpha)\eta(X)\eta(W) \\ & - (\beta + \alpha + \lambda)g(X, W) - \alpha g(\phi X, W)\} \\ & + 2\alpha g(\phi Z, X)\{3\beta + \alpha)\eta(Y)\eta(W) \\ & - (\beta + \alpha + \lambda)g(Y, W) - \alpha g(\phi Y, W)\}. \end{aligned} \tag{6.10}$$

Putting $X = Z = e_i$ in (6.10), we get

$$0 = (3\beta + \alpha)(\beta - \alpha - 2\lambda - \tfrac{r}{2})\{g(Y, W) - 3\eta(Y)\eta(W)\} \tag{6.11}$$

$$- \frac{2\alpha}{n-1}\{\beta + \alpha + \lambda)g(W, \phi Y) - \alpha(g(Y, W) - \eta(Y)\eta(W).$$

Taking $Y = \xi$ in (6.11), we, have

$$0 = (3\beta + \alpha)(\beta - \alpha - 2\lambda - \tfrac{r}{2}).$$

that is, either

$$\alpha = -3\beta,$$

or,

$$\lambda = \frac{\beta - \alpha - \frac{r}{2}}{2}$$

Thus we say the follwong:

Theorem 6. *Let M be a three-dimensional normal almost contact metric manifold admitting quarter symmetric non-metric connection which admits the Ricci soliton for $V = \xi$. If $\widetilde{P}(X, Y)\cdot S = 0$, then either $\lambda = \frac{\beta - \alpha - \frac{r}{2}}{2}$ or, $\alpha = -3\beta$, provided α, β = constant.*

Let $\widetilde{Z}(X,Y)\cdot\widetilde{S} = 0$. Then

$$(\widetilde{Z}(X,Y)\widetilde{S})(V,W) = -\widetilde{S}(\widetilde{Z}(X,Y)V,W) - \widetilde{S}(V,\widetilde{Z}(X,Y)W) = 0.$$

Using (6.6) and (6.1) in the above equation, we obtain

$$0 = (3\beta+\alpha)\left(\beta - \alpha - 2\lambda - \frac{r}{2} - \frac{r}{n(n-1)}\right)\{g(Y,V)\eta(X)\eta(W) \tag{6.12}$$

$$- g(X, V)\eta(Y)\eta(W) + g(Y, W)\eta(X)\eta(V) - g(X, W)\eta(Y)\eta(V)\}$$

$$- 2\alpha\{\alpha + 3\beta\}\{g(\phi Y, V)\eta(X)\eta(W) - g(\phi X, V)\eta(Y)\eta(W)\}.$$

Putting $X = V = e_i$ in (6.12), we get

$$0 = (3\beta+\alpha)\left(\beta-\alpha-2\lambda-\frac{r}{2}-\frac{r}{n(n-1)}\right)\{g(Y,W)-3\eta(Y)\eta(W)\}. \tag{6.13}$$

Putting $Y = \xi$ in (6.13), we have

$$0 = (3\beta+\alpha)\left(\beta-\alpha-2\lambda-\frac{r}{2}-\frac{r}{n(n-1)}\right)$$

that is, either

$$\alpha = -3\beta,$$

or,

$$\lambda = \frac{\beta-\alpha-\frac{r}{2}-\frac{r}{n(n-1)}}{2}$$

Thus we give the followng:

Theorem 7. *Let M be a three-dimensional normal almost contact metric manifold admitting quarter symmetric non-metric connection which admits the Ricci soliton for* $V = \xi$. *If* $\tilde{Z}(X, Y)\cdot\tilde{S} = 0$, *then either* $\lambda = \dfrac{\beta-\alpha-\frac{r}{2}-\frac{r}{n(n-1)}}{2}$ *or,* $\alpha = -3\beta$, *provided* α, β = *constant*

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ON SOME CLASSES OF NEARLY KENMOTSU MANIFOLDS

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ABSTRACT : The object of the present paper is to study second order recurent tensor fields in nearly Kenmotsu manifolds and improve the proof of a Theorem of Najafi et al. Beside this we study Ricci semisymmetric nearly Kenmotsu manifolds and obtain some equivalent conditions. Finally, we consider semiparallel, 2-semiparallel and pseudo-parallel invariant submanifolds of nearly Kenmotsu manifolds.

Key words and Phrases : Nearly Kenmotsu manifold, semiparallel, 2-semiparallel, pseudo-parallel, invariant submanifolds.

AMS Classification : 53C15, 53C25.

1. INTRODUCTION

Among Riemanian manifolds, the most interesting and most important for applications are the symmetric ones. From the local point of view they were introduced independently by Shirokov [26] and Levy [16] as a Riemannian manifold with covariant constant curvature tensor R, i.e., with

$$\nabla R = 0.$$

where ∇ is the Levi-Civita connection. An extensive theorey of symmetric Riemannian manifolds was worked out by Cartan in 1927.

Later, a similar development took place in the geometry of submanifolds in the space forms, where a fundamental role is played by metric tensor g (as the induced Riemannian metric) and the second fundamental form α. Besides the Levi-Civita connection ∇, with $\nabla g = 0$, a normal connection $\nabla^{\perp}$ is also defined. The submanifolds with parallel second fundamental form, i.e., with

$$\overline{\nabla} \alpha = 0.$$

where $\overline{\nabla}$ is the pair of ∇ and $\nabla^{\perp}$, deserve special attention. The theory of parallel submanifolds is concisely treated in recent monograph by Lumiste [17].

As a generalization of symmetric manifolds Cartan in 1946 introduced the notion of semisymmetric manifolds. A Riemannian manifold is called semisymmetric if the curvature tensor R satisfies $R(X, Y). R = 0$, where $R(X, Y)$ is considered as a field of linear operators, acting on R. A Riemannian manifold is said to be Ricci semisymmetric if $R(X, Y).S = 0$, where S is the Ricci tensor of type (0, 2).

Semisymmetric manifolds were classified by Szabo, locally in [27]. The classification results of Szabo were presented in the book [5].

Parallel submanifolds were likewise later placed in a more general class of sub-manifolds generalizing the notion of parallel submanifolds.

Let M and $\tilde{M}$ be two Riemannian or semi-Riemannian manifolds, if : $M \to \tilde{M}$ an isometric immersion, α the second fundamental form and $\overline{\nabla}$ the van der Waerden-Bortolotti connection of M. An immersion is said to be *semiparallel* if

$$\overline{R}(X,Y)\alpha = (\overline{\nabla}_X \overline{\nabla}_Y - \overline{\nabla}_Y \overline{\nabla}_X - \overline{\nabla}_{(X,Y)})\, \alpha = 0 \tag{1.1}$$

holds for all vector fields tangent to M[8], where $\overline{R}$ denots the curvature tensor of the connection $\overline{\nabla}$. Semiparallel immersion have been studied by many authors, see, for example, [9, 10, 11, 15, 17].

In [2] Arslan et al defined and studied submanifolds satisfying the condition

$$\overline{R}(X,Y), \overline{\nabla}_{\alpha} = 0 \tag{1.2}$$

for all vector fields. X, Y tangent to M. Submanifolds satisfying (1.2) are called *2-semiparallel.*

An immersion is said to be pseudo-parallel [3, 4] if

$$\overline{R}(X,Y).\alpha = fQ(g, \alpha) \tag{1.3}$$

holds for all vector fields X, Y tangent to M, where f denotes real valued function on M and for a $(0, k)$, $k \geq 1$ tensor T and a (0, 2) tensor E, $Q(E,T)$ is defined by [28]

$$Q(E,\ T)(X_1,\ X_2 \ldots\ldots,\ X_k;\ X,\ Y) = -T((X \wedge_E Y)\, X_1,\ X_2, \ldots\ X_k) \tag{1.4}$$
$$-T(X_1,\ (X \wedge_E Y)\, X_2, \ldots\ X_k)$$
$$-\ldots -T(X_1,\ X_2,\ \ldots\ldots X_{k-1},\ (X \wedge_E Y)\, X_k).$$

where $X \wedge_E Y$ is defined by

$$(X,\ \wedge_E Y)\, Z = E(Y,\ Z)X - E(X,\ Z)Y. \tag{1.5}$$

Therefore we can take the following equation instead of the equation (1.1)

$$R^{\perp}(X,\ Y)\ \alpha\ (U,\ V) - \alpha(R(X,\ Y)U,\ V) - \alpha(U,\ R(X,\ Y)V) \tag{1.6}$$
$$= -f[\alpha(X \wedge_g Y)U,\ V) + \alpha(U,\ (X \wedge_g Y)V)].$$

In a recent paper [19], *Ozgur* and Murathan studied semiparallel and 2-semiparallel invariant submanifolds of Lorentzin para-Sasakian manifolds. Also in [18] V. Mangione studied invariant submanifolds of Kenmotsu manifolds. In a recent paper Najafi and Kashani [20] studied second order symmetric closed recurrent tensor in a nearly Kenmotsu manifold under certain conditions. A (0, 2)-tensor field α on a Riemannain manifold $(M,\ g)$ is said to be a recurrent tensor if T satisfies $\nabla T = \lambda \otimes T$ for some 1-form λ. The 1-form λ is called the recurrence covector of T. The set of closed recurrent tensors contains the set of parallel tensors ($\lambda = 0$) as a subset. In the present paper we improve the proof of the Theorem 3.1 of [20].

Motivated by these studies of the above authors [18, 19], in this present paper we consider invariant submanifolds of nearly Kenmotsu manifolds. We consider the semiparallel, 2-semiparallel, pseudo-parallel invariant submanifolds of nearly Kenmotsu manifolds. The paper is organized as follows :

In section 2, we give necessary details about submanifolds which will be used for latter sections. In section 3, we give a brief account of nearly Kenmotsu manifolds and their submanifolds. In section 4, we study second order recurrent tensor fields in nearly Kenmotsu manifolds and improve the proof of the Theorem 3.1 of [20] without assuming any condition. Next we prove that there does not exist any non-zero parallel 2-form in nearly Kenmotsu manifolds. In section 5

we consider Ricci semisymmetric nearly Kenmotsu manifolds and we obtain some equivalent conditions. In section 6, give some preliminaries about invariant submanifold of nearly Kenmotsu manifolds and prove a Propostion. Finally, we study semiparallel, 2-semiparallel, pseudo-parallel invariant submanifolds of nearly Kenmotsu manifolds respectively. As a consequence of the above results we obtain some important corollaries.

2. BASIC CONCEPTS

Let (M, g) be a n-dimensional Riemannian submanifold of an $(n+d)$-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. We denote by $\overline{\nabla}$ and ∇ the Levi-Civita connection of $\tilde{M}$ and M, respectively. Then we have the Gauss and Weingarten formulas

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha(X, Y). \tag{2.1}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}{}_X N.$$

where X, Y are the vector fields tangent to M and N is a normal vector field on M, respectively. $\nabla^{\perp}$ is called the *normal connection* of M. We call α the *second fundamental form* of submanifolds M. If $\alpha = 0$ then the manifold is said to be *totally geodestic.* For the second fundamental form α, the covariant derivative of α is defined by

$$(\overline{\nabla}_X \alpha)(Y, Z) = \nabla^{\perp}{}_X(\alpha(Y, Z) - \alpha(\nabla_X, Z) - \alpha(Y, \nabla_X Z) \tag{2.2}$$

for any vector fields X, Y, Z tangent M. Then $\overline{\nabla}\alpha$ is a normal bundle valued tensor of type (0.3) and is called the *third fundamental form* of M. $\overline{\nabla}$ is called the *vander waerden-Bortolotti connection* of M, i.e., $\overline{\nabla}$ is the connection in $TM \oplus T^{\perp}M$ built with ∇ and $\nabla^{\perp}$. If $\overline{\nabla}\alpha = 0$, then M is said to have *parallel second fundamental* form [7]. From the Gauss and Weingarten formulas we obtain

$$(\overline{R}(X,Y)Z)^T = R(X,Y)Z + A_{\alpha(X, Z)}Y - A_{\alpha(Y, Z)}X. \tag{2.3}$$

Therefore from (1.1) we have

$$(\overline{R}(X,Y).\alpha)(U, V) = R^{\perp}(X,Y)\,\alpha(U,V) - \alpha(R(X,Y)U,V) - \alpha(U, R(X,Y)V) \tag{2.4}$$

for all vector fields X, Y, U and V tangent to M, where

$$R^{\perp}(X,Y) = [\nabla^{\perp}_{X}, \nabla^{\perp}_{Y}] - \nabla^{\perp}_{[X,Y]} \tag{2.5}$$

and $\overline{R}$ denotes the curvature tensors of $\overline{\nabla}$. Similarly

$$\begin{aligned}(\overline{R}(X,Y).\nabla\alpha)(U, V, W) = R^{\perp}(X, Y)(\overline{\nabla}\alpha)(U, V, W) \\ - (\overline{\nabla}\alpha)(R(X, Y)U, V, W) - (\overline{\nabla}\alpha)(U, R(X, Y)V, W) \\ - (\overline{\nabla}\alpha)(U, V, R(X, Y)W)\end{aligned} \tag{2.6}$$

for all vector fields X, Y, U, V and W tangent to M, where $(\overline{\nabla}\alpha)(U, V, W) = (\overline{\nabla}_{U}\alpha)(V, W)$ [2].

3. NEARLY KENMOTSU MANIFOLDS

An almost contact metric manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ is called a nearly Kenmotsu manifold by Shukla [25] if the following relation holds :

$$(\overline{\nabla}_{X}\phi)Y + (\overline{\nabla}_{Y}\phi)X = -\eta(Y)\phi X - \eta(X)\phi Y. \tag{3.1}$$

where ∇ is the Levi-Civita connection of g. Moreoever, if M satisfies

$$(\overline{\nabla}_{X}\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X. \tag{3.2}$$

then it is called a Kenmotsu manifold [14, 21]. It is easy to see that every Kenmotsu manifold is a nearly Kenmotsu manifold, but the converse is not true. A nearly Kenmotsu manifold is not a K-contact manifold and hence is not a Sasakian manifold [24]. Recently, nearly Kenmotsu manifolds have been studied extensively, see [12, 13].

Tensor algebras play a prominent role in differential geometry, in particular in Riemannian geometry. Wong [29] studied recurrent tensor fields on a manifold endowed with a linear connection. Levy showed that on a space of constant curvature, second-order symmetric parallel nonsingular tensors are constant multiples of the metric tensor [16].

First, we recall some important identities holding in every n-dimensional nearly Kenmotsu manifold (M, ϕ, ξ, η, g) (for more details, see [13]) :

$$\phi\xi = 0,\ \eta(\xi) = 1\ \phi^{2}X = -X + \eta(X)\xi,\ \eta \circ \phi = 0, \tag{3.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \; g(X, \xi) = \eta(X), \tag{3.4}$$

$$g(X, \phi Y) = -g(\phi X, Y), \; (\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{3.5}$$

$$R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X, \; R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{3.6}$$

$$S(\phi X, \phi Y) = S(X, Y) - (1-n)\eta(X)\eta(Y), \tag{3.7}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{3.8}$$

where R is the Riemannian curvature and S is the Ricci tensor of g.

In a nearly Kenmotsu manifold (M, ϕ, ξ, η, g) the following relations hold [13];

$$\nabla_X \xi = X - \eta(X)\xi, \; \nabla_\xi \xi = 0. \tag{3.9}$$

Hence, the orbits of ξ are godesics.

Example 3.1. [6] *If N_1, N_2 are two nearly cosymplectic manifolds then their product is a nearly Kahler manifold, hence $R \times_f (N_1 \times N_2)$ is a nearly Kenmotsu manifold.*

Example 3.2. [22] *The sphere S^6 endowed with the usual almost Hermitian structure is a nearly Kahler manifold, so that $R \times_f S^6$ is nearly Kenmotsu manifold.*

4. NEARLY KENMOTSU MANIFOLDS WITH SECOND ORDER RECURRENT TENSOR FIELDS

Suppose (M, ϕ, ξ, η, g) is a nearly Kenmotsu manifold and T is a recurrent (0.2) tensor on M, therefore

$$(\nabla_Z T)(X, Y) = \lambda(Z)T(X, Y), \tag{4.1}$$

where λ is a 1-form called recurrent covector of T.

Let us suppose that $T \neq 0$ and Q be the (1.1) tensor defined by

$$g(QX, Y) = T(X, Y), \tag{4.2}$$

Hence (4.1) can be written as

$$(\nabla_Z Q)X = \lambda(Z)Q(X). \tag{4.3}$$

We now define a function f on M by

$$f^2 = g(Q, Q). \tag{4.4}$$

Using the fact that $\nabla_W g = 0$, it follows from (4.4) that

$$2f(Wf) = f^2 (\lambda(W)). \tag{4.5}$$

Since $f \neq 0$, from (4.5) we get

$$(Wf) = f\,\lambda(W). \tag{4.6}$$

From (4.6) we have

$$Z(Wf) = \frac{1}{f}(Zf)(Wf) + (Z\,\lambda(W))f. \tag{4.7}$$

Hence,

$$Z(Wf) - W(Zf) = \{Z\lambda(Y) - W\lambda(Z)\}f. \tag{4.8}$$

Therefore, we get

$$(\nabla_Z\nabla_W - \nabla_W\nabla_Z - \nabla_{[Z,W]})f = \{Z\lambda(W) - W\lambda(Z) - \lambda([Z, W])\}f. \tag{4.9}$$

Since the left hand side of (4.9) is identically zero and f is non-zero on M by our assumption, we obtain

$$d\lambda(Z, W) = 0. \tag{4.10}$$

i.e., the 1-form λ is closed.

Now from $(\nabla_W Q)X = \lambda(W)\,QX$, we get

$$\begin{aligned}(\nabla_z\nabla_w Q)X - (\nabla_{\nabla ZW} Q)X &= [(\nabla_Z)W + \lambda(Z)\,\lambda(W)]QX \\ &= 2d\lambda(Z, W)QX \\ &= 0.\end{aligned} \tag{4.11}$$

Therefore,

$$(R(W, Z).Q)(X) = 0. \tag{4.12}$$

where $R(W, Z)$ is to be considered as a derivation of the tensor algebra at each point of the manifold for all tangent vectors W and Z. Hence

$$(R(W, Z).T)(X, Y) = 0. \tag{4.13}$$

This implies

$$T(R(W, Z)X, Y) + T(X, R(W, Z)Y = 0. \tag{4.14}$$

Putting $X = Y = W = \xi$ in (4.14), we have

$$T(R(\xi, Z)\xi,\xi) + T(\xi, R(\xi, Z)\xi) = 0. \tag{4.15}$$

Using (3.4) and (3.6) in (4.15) we have

$$2g(X, \xi)T(\xi,\xi) - T(X, \xi) - T(\xi, X) = 0. \tag{4.16}$$

Differentiating (4.16) along Y and using $\nabla_\xi \xi = 0$ yields

$$2\{g(\nabla_Y X, \xi) + g(X, \nabla_Y \xi)\} T(\xi,\xi) + 2g(X, \xi) \nabla_Y T(\xi,\xi) = T(\nabla YX.\xi)$$

$$+ T(X, \nabla_Y \xi) + T(\nabla Y\xi, X) + T(\xi, \nabla YX). \tag{4.17}$$

Putting $X = \nabla_Y X$ in (4.16) we have

$$2g(\nabla_Y X, \xi)T(\xi,\xi) - T(\nabla_T X,\xi) - T(\xi, \nabla_Y X) = 0. \tag{4.18}$$

Therefore from (4.9) and (4.11) we obtain

$$2g(X, \nabla_Y \xi)\, T(\xi,\xi) + 2g(X, \xi)\nabla_Y T(\xi,\xi) = T(X, \nabla_Y \xi) + T(\nabla_Y \xi, X). \tag{4.19}$$

Using (3.9) in (4.19) we have

$$2g(X, Y)T(\xi,\xi) + 2g(X,\xi)T(\nabla_Y \xi,\xi) = T(X, Y) + T(Y, X). \tag{4.20}$$

Since T is symmetric, then from (4.20) we have

$$2g(X, Y)T(\xi,\xi) + 2g(X,\xi)T(\nabla_Y \xi,\xi) = 2T(X, Y). \tag{4.21}$$

It follows from (4.21)

$$g(X, Y)T(\xi,\xi) + \eta(X)T(Y,\xi) - \eta(X)\eta(Y)T(\xi,\xi) = T(X, Y). \tag{4.22}$$

Putting $Y = \xi$ in (4.22) we get

$$T(X, \xi) = \eta(X)T(\xi,\xi). \tag{4.23}$$

Using (4.23) in (4.22) we have

$$T(X, Y) = g(X, Y)T(\xi,\xi). \tag{4.24}$$

Thus in view of the above discussions we can state the following :

Theorem 4.1. *On a nearly Kenmotsu manifold a second order symmetric recurrent tensor is a constant multiple of the metric tensor g.*

Thus we prove the above Theorem without assuming any condition. Hence we improve the proof of the Theorem 3.1 of [20].

Suppose that T is a non zero parallel 2-form. That is, T is a skew symmetric tensor of type (0,2). Then we have

$$T(\xi, \xi) = 0. \tag{4.25}$$

This implies

$$T(\nabla_X \xi, \xi) = 0. \tag{4.26}$$

Using (6.1) in (4.26) we have

$$T(X, \xi) - \eta(X)T(\xi,\xi) = 0. \tag{4.27}$$

In view of (4.25), (4.27) gives us

$$T(X, \xi) = 0. \tag{4.28}$$

Putting $X = \nabla_Y X$ in (4.28) we have

$$T(\nabla_Y X, \xi) = 0. \tag{4.29}$$

Again from (4.28) we have

$$T(\nabla_Y X,\xi) + T(X, \nabla_Y\xi) = 0. \tag{4.30}$$

This implies

$$\begin{aligned} T(X, Y) &= \eta(Y)T(X, \xi) \\ &= \eta(Y)\eta(X)T(\xi, \xi) \\ &= 0. \end{aligned} \tag{4.31}$$

which is a contradiction. Hence we can state the following :

Theorem 4.2. *There does not exist any non-zero parallel 2-form in a nearly Kenmotsu manifold.*

5. NEARLY KENMOTSU MANIFOLD SATISFYING $R(X, Y).S = 0$

In this section we prove the following equivalent conditions :

Theorem 5.1. *For a nearly Kenmotsu manifold the followingconditions are equivalent :*

(1) *M is an Einstein manifold.*

(2) $\nabla S = 0$

(3) *$R(X, Y).S = 0$ for all X, Y*

(4) *$R(\xi, Y).S = 0$ for all Y,*

where S denotes the Ricci tensor.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is clear.

Now we assume condition (4) which is equivalent to

$$S(R(\xi, Y)U, V) + S(U, R(\xi, Y)V) = 0. \tag{5.1}$$

Putting $X = \xi$, $V = \xi$ in (5.1) we get

$$S(R(\xi, Y)U, \xi) + S(U, R(\xi, Y)\xi) = 0, \tag{5.2}$$

This implies

$$S(-g(Y, U)\xi + \eta(U)Y, \xi) + S(U, Y - \eta(Y)\xi) = 0. \tag{5.3}$$

Using the condition $S(X, \xi) = -2\eta\eta(X)$ in (5.3) we get

$$2ng(Y, U) - 2\eta\eta(U)\eta(Y) + S(U, Y) + 2\eta\eta(Y)\eta(U) = 0. \tag{5.4}$$

This implies

$$S(U, Y) = -2ng(Y, U). \tag{5.5}$$

Therefore M is an Einstein manifold.

6. INVARIANT SUBMANIFOLD OF NEARLY KENMOTSU MANIFOLDS

A submanifold M of a nearly Kenmotsu manifold $\tilde{M}$ is called an invariant submanifold of $\tilde{M}$ if $\phi(TM)\subset(TM)$. We now prove the following proposition which will be used in the next section :

Proposition 6.1. *Let M^n be an invariant submanifold of a nearly Kenmotsu manifold $\tilde{M}$. Then the following equalities holds on M^n :*

$$\nabla x\xi = X - \eta(X)\xi, \tag{6.1}$$

$$\alpha(X, \xi) = 0. \tag{6.2}$$

$$(\nabla x\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X. \tag{6.3}$$

$$\bar{R}(X, Y)\xi = R(X, Y)\xi \tag{6.4}$$

$$\alpha(X, \phi Y) = \phi\alpha(X, Y). \tag{6.5}$$

for all vector fields X, Y tangent to M.

Proof. Since M^n is an invariant submanifold of a nearly Kenmotsu manifold $\tilde{M}$

$$\bar{\nabla}_X\xi = X - \eta(X)\xi. \tag{6.6}$$

Using the Gauss formula (2.1) we get

$$X - \eta(X)\xi = \nabla_X\xi = \nabla_X\xi + \alpha(X, \xi), \tag{6.7}$$

which gives us

$$\nabla_X\xi = X - \eta(X)\xi.$$

$$\alpha(X, \xi) = 0.$$

so we get (6.1) and (6.3).

Since $\tilde{M}$ is a nearly Kenmotsu manifold, we get from (3.2)

$$(\overline{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X. \tag{6.8}$$

Then in view of the Gauss formula, we have

$$(\overline{\nabla}_X\phi)Y = \nabla_{X_\phi}Y + \alpha(X, \phi Y) - \phi\nabla_X Y - \phi\alpha(X, Y). \tag{6.9}$$

Comparing tangential and normal part of (6.8) and (6.9), we get

$$(\nabla_X\phi)Y = g(\phi X, Y)\xi - \eta(Y)\,\phi X.$$

$$\alpha(X, \phi Y) = \phi\alpha(X, Y),$$

So we obtain (6.3) and (6.5) From the Gauss equation (2.3) we have

$$\overline{R}(X,Y)\xi = R(X, Y)\xi + A_{\alpha(x,\xi)}Y - A_{\alpha(Y,\xi)}X. \tag{6.10}$$

Then using $\alpha(X, \xi) = 0$ we find

$$\overline{R}(X,Y)\xi = R(X, Y)\xi, \tag{6.11}$$

which in view of (3.6), gives (6.4).

Therefore from the Proposition 6.1 we can state the following :

Theorem 6.1. *An invariant submanifold M^n of a nearly Kenmotsu manifold $\tilde{M}$ is a nearly Kenmotsu manifold.*

7. SEMIPARALLEL, 2-SEMIPARALLEL AND PSEUDO-PARALLEL INVARIANT SUBMANIFOLDS OF NEARLY KENMOTSU MANIFOLDS

In this section we consider semiparallel, 2-semiparallel and pseudo-parallel in-variant submanifolds M of a nearly Kenmotsu manifolds $\tilde{M}$. Now we prove the following Theorems :

Theorem 7.1. *Let M^n be an invariant submanifold of a nearly Kenmotsu manifold $\tilde{M}$. Then M^n is semiparallel if and only if M^n is totally geodesic.*

Proof. Since M^n is semiparallel, we have $\overline{R}.\alpha = 0$. Then from (2.4) we have

$$R^{\perp}(X, Y)\,\alpha(U, V) - \alpha(R(X, Y)U, V) - \alpha(U, R(X, Y)V) = 0 \tag{7.1}$$

for all vector fields X, Y, U and V tangent to M. Putting $X = V = \xi$ in (7.1) we obtain

$$R^{\perp}(\xi, Y)\,\alpha(U, \xi) - \alpha(R(\xi, Y)U, \xi) - \alpha(U, R(\xi, Y)\xi) = 0 \tag{7.2}$$

Therefore (7.2) and (6.2) yields

$$\alpha(U, R(\xi, Y)\xi) = 0. \tag{7.3}$$

Then with the help of (6.4) we have

$$\alpha(Y, U) = 0. \tag{7.4}$$

which implies that M^n is totally geodesic. On the other hand it is easy to see that if M^n is totally geodesic, then it is semiparallel. This completes the proof of the Theorem.

Now, if M^n has parallel second fundamental form, then it is semiparallel. Therefore we can state the following :

Corollary 7.1. *Let M^n be an invariant submanifold of a nearly Kenmotsu manifold $\widetilde{M}$. Then M^n has parallel second fundamental form if and only if M^n is totally geodesic.*

The second fundamental form α is birecurrent [23] if there exists a non-zero covariant tensor field B such that $(\nabla_X\nabla_W\alpha - \nabla_{\nabla_X W}\alpha)(Y, Z) = B(X, W)\,\alpha(Y, Z)$. In a recent paper [1] Aikawa and Matsuyama proved that if a tensor field T is birecurrent, then $R(X, Y).T = 0$. Therefore by virture of the Theorem 6.1 we can state the following :

Corollary 7.2. *Let M^n be an invariant submanifold of a nearly Kenmotsu manifold $\widetilde{M}$. Then M^n has birecurrent second fundamental form if and only if M^n is totally geodesic.*

Theorem 7.2. *Let M^n be an invariant submanifold of a nearly Kenmotsu manifold $\widetilde{M}$. Then M^n is 2-semiparallel if and only if M^n is totally geodesic.*

Proof. Since M^n is 2-semiparallel. Therefore we have $\overline{R}.\overline{\nabla}\alpha = 0$. Then from (2.6) we obtain

$$R^{\perp}(X, Y)(\overline{\nabla}\alpha)(U, V, W) - (\overline{\nabla}\alpha)(R(X, Y)\,U, V, W) \tag{7.5}$$

$$- (\overline{\nabla}\alpha)(U,\ R(X,\ Y)V,\ W) - (\overline{\nabla}\alpha)(U,\ V,\ R(X,\ Y)W) = 0$$

for all vector fields $X,\ Y,\ U,\ V$ and W tangent to M. Taking $X = V = \xi$ in (7.5) we obtain

$$R^{\perp}(\xi,\ Y)(\overline{\nabla}\alpha)(U,\ \xi,\ W) - (\overline{\nabla}\alpha)(R(\xi,\ Y)\ U,\ \xi,\ W) \tag{7.6}$$
$$- (\overline{\nabla}\alpha)(U,\ R(\xi,\ Y)\xi,\ W) - (\overline{\nabla}\alpha)(U,\ \xi,\ R(\xi,\ Y)W) = 0$$

Therefore in view of (6.1), (6.2) and (6.5) we have the following equalities:

$$\begin{aligned}(\overline{\nabla}\alpha)(U,\ \xi,\ W) &= (\overline{\nabla}_U\alpha)\ (\xi,\ W) \\ &= \nabla^{\perp}_U(\alpha(\xi,,W)) - \alpha(\nabla_U\xi, W) - \alpha(\xi, \nabla_U W) \\ &= -\alpha(U - \eta(U)\xi,\ W) \\ &= -\alpha(\phi U,\ W).\end{aligned} \tag{7.7}$$

$$\begin{aligned}(\overline{\nabla}\alpha)(R(\xi,\ Y)U,\xi,W) &= (\overline{\nabla}_{R(\xi,Y)U}\alpha)\ (\xi,\ W) \\ &= \nabla^{\perp}_{R(\xi,Y)U}(\alpha(\xi,\ W)) - \alpha(_{R(\xi,Y)U}\xi,\ W) \\ &\quad -\alpha(\xi,\ \nabla_{R(\xi,Y)U}W) \\ &= -\alpha(\nabla_{R(\xi,Y)U}\xi,\ W) \\ &= -\ \alpha(R(\xi,\ Y)U - \eta(R(\xi,\ Y)U)\xi,\ W)\end{aligned} \tag{7.8}$$

$$\begin{aligned}(\overline{\nabla}\alpha)(U,R(\xi,\ Y)\xi,W &= (\overline{\nabla}_U\alpha)(R(\xi,\ Y)\xi,\ W) \\ &= \nabla^{\perp}_U\alpha(R(\xi,\ Y)\xi,\ W) - \alpha(\nabla_U R(\xi,Y)\xi,\ W) \\ &\quad -\alpha(R(\xi,\ Y)\xi,\ \nabla_U W) \\ &= \nabla^{\perp}_U\alpha(Y,W) - \alpha(\nabla_U(Y - \eta)(Y)\xi),\ W) \\ &\quad -\alpha(Y,\ \nabla_U W)\end{aligned} \tag{7.9}$$

and

$$\begin{aligned}(\overline{\nabla}\alpha)(U,\xi,R(\xi,\ Y)W &= (\overline{\nabla}_U\alpha)(\xi,\ R)\xi,\ Y)W) \\ &= \nabla^{\perp}_U\alpha(\xi,\ R(\xi,\ Y)W) - \alpha(\nabla_U\xi,\ R(\xi,\ Y)W)\end{aligned} \tag{7.10}$$

$$-\alpha(\xi, \nabla_U R(\xi, Y)W),$$

$$= -\alpha(U - \eta(U)\xi, R(\xi, Y)W)$$

Using (7.7) – (7.10) in (7.6), we obtain

$$R^{\perp}(\xi, Y)\alpha(U, W) + \alpha(\phi R(\xi, Y)U, W) - \nabla^{\perp}_U\alpha(Y, W) \tag{7.11}$$

$$+ \alpha(\nabla_U(Y - \eta(Y)\xi, W) + \alpha(Y, \nabla_U W) - \alpha(U - \eta(U)\xi, R(\xi, Y)W) = 0.$$

Putting $W = \xi$ in (7.11) we have

$$\alpha(Y, U) + \alpha(Y, U) = 0. \tag{7.12}$$

It follows that

$$2\alpha(Y, U) = 0. \tag{7.13}$$

Hence

$$\alpha(Y, U) = 0. \tag{7.14}$$

Therefore M^n is totally geodesic. Conversely, if M^n is totally geodesic ($\alpha = 0$), then $\overline{R}\cdot\overline{\nabla}_\alpha = 0$ is trivial. Hence the Theorem.

Theorem 7.3. *Let M^n be an invariant submanifold of a nearly Kenmotsu manifold $\widetilde{M}$. Then M^n is pseudo-parallel if and only if M^n is totally geodesic provided $f \neq -1$.*

Proof. Since M^n is pseudo-parallel invariant submanifolds of nearly Kenmotsu manifolds. Therefore we have

$$\overline{R}\,(X, Y).\alpha = fQ(g, \alpha) \tag{7.15}$$

holds for all vector fields X, Y tangent to M, where f denotes real valued function on M^n. The equation (7.15) can be written as

$$R^{\perp}(X, Y)\alpha(U, V) - \alpha(R(X, Y)U, V) - \alpha(U, R(X, Y)V) \tag{7.16}$$

$$= -f[\alpha(X \wedge_g Y)U, V) + \alpha(U, (X, \wedge_g Y)V)].$$

where $(X \wedge_g Y)$ is defined by

$$(X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y. \tag{7.17}$$

Using (7.17) in (7.16) we have

$$R^{\perp}(X, Y)\alpha(U, V) - \alpha(R(X, Y)U, V) - \alpha(U, R(X, Y)V) \tag{7.18}$$
$$= -f[g(Y, U)\alpha(X, V) - g(X, U)\,\alpha(Y, V)$$
$$+g(Y, U)\,\alpha(U, X) - g(X, V)\,\alpha(U, Y)].$$

Putting $X = V = \xi$ in (7.18) we get

$$R^{\perp}(\xi, Y)\,\alpha(U, \xi) - \alpha(R(\xi, Y)U, \xi) - \alpha(U, R(\xi, Y)\xi) \tag{7.19}$$
$$= -f[g(Y, U)\alpha(\xi, V\xi) - g(\xi, U)\,\alpha(Y, \xi)$$
$$+g(Y, \xi)\,\alpha(U, \xi) - g(\xi, \xi)\,\alpha(U, Y)].$$

It follows that

$$(f + 1)\,\alpha(U, Y) = 0 \tag{7.21}$$

i.e., $\alpha(U, Y) = 0$ which gives M^n it totally geodesic, provided $f \neq -1$.

Conversely, let M^n be totally geodesic. Clearly from (6.4) it follows that M^n is totally geodesic. This completes the proof of the Theorem.

In view of the Theorem 6.1 – 6.3, we can state the following :

Theorem 7.4. *Let M^n be an invariant submanifold of a nearly Kenmotsu manifold $\tilde{M}$. Then the following statements are equivalent :*

(1) *M^n is semiparallel;*

(2) *M^n is 2-semiparallel*

(3) *M^n is pseudo-parallel provided $f \neq -1$.*

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DERIVATIVE OF $e^{A(x)}$

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ABSTRACT : The derivative of $\exp[A(x)]$ with respect to x has been determined, where $A(.) = (a_{ij}(.))$ *is* an $n \times n$ matrix function, defined on an interval J, i.e., each a_{ij} is a real valued differentiable function on J, keeping in mind that $A(x)$, $A'(x)$, for $x \in J$, may or may not commute ($'$ denoting differentiation with respect to x).

Key words : Derivative, Matrix function, Exponential of a matrix.

1. INTRODUCTION

Let $A(.) = (a_{ij}.)$ be an $n \times n$ matrix function, defined on an interval J, where each a_{ij} is a real-valued function on J.

The exponential function $e^{A()}$ is defined as

$$e^{A(x)} = I_n + \sum_{r=1}^{\infty} \frac{1}{r!} A^r(x), \ (x \in J), \tag{1.1}$$

where I_n denotes the $n \times n$ identity matrix. That, the infinite series in (1.1) is uniformly convergent on J, can be proved by taking any norm of $A(x)$, since

$$\sum_{r=0}^{\infty} \left(\frac{I}{r!}\right) y^r \ \textit{is convergent for all real } y.$$

Problem : *To find* $\frac{d}{dx} e^{A(x)}$ for $x \in J$, *if the entries* $a_{ij}(.)$ *are all differentiable on* J. If $n = 1$, it is well-known that

$$\frac{d}{dx} e^{A(x)} = e^{A(x)} \ A'(x) = A'(x) e^{A(x)}, \tag{1.2}$$

where $'$ denotes differentiation with respect to x.

If $n > 1$, the validity of (1.2) is not guaranteed; (1.2) is not true if $A(x)A'(x) \neq A'(x)A(x)$.

The aim here is to establish the formula for $\frac{d}{dx}e^{A(x)}$, that will be true for all possible cases.

The following theorem represents the result sought for.

Theorem : If A is an $n \times n$ matrix on the set of real valued differentiable functions on an interval J, then

$$\frac{d}{dx}e^{A(x)} = e^{A(x)}.A'(x) + \sum_{n=0}^{\infty}\frac{I}{(n+2)!}\sum_{k=1}^{n+1}A^{n+1-k}C(A', A^k), \tag{1.3}$$

$$= A'(x).e^{A(x)} + \sum_{n=0}^{\infty}\frac{I}{(n+2)!}\sum_{k=1}^{n+1}\ C(A^k, A')A^{n+1-k}, \tag{1.4}$$

where $C(A,\ B) := AB - BA$.

Both the formulae (1.3) and (1.4) are valid whether or not $A(x)$, $A'(x)$ commute with each other.

Taking any norm on the set of matrix functions, it is noted that

$$\left\|\sum_{k=1}^{n+1}A^{n+1-k}C(A', A^k)\right\| \leq \sum_{k=1}^{n+1}\|A\|^{n+1-k}\left\|C(A', A^k)\right\| \leq 2(n+1)\|A'\|.\|A\|^{n+1}, \tag{1.5}$$

and

$$\left\|\sum_{k=1}^{n+1}C(A^k, A')A^{n+1-k}\right\| \leq \sum_{k=1}^{n+1}\left\|C(A^k, A')\right\|\|A\|^{n+1-k} \leq 2(n+1)\|A'\|.\|A\|^{n+1}, \tag{1.6}$$

(1.5) / (1.6) respectively ensures that the series in (1.3) / (1.4) is convergent.

2. Derivative of A(x) B(x), x ∈ J

Let $A(.) = (a_{ij}(.))$ and $B(.) = (b_{ij}(.))$ be two $n \times n$ matrices with entries that are real valued differentiable functions on an interval J.

Then, for all $x \in J$, $A(x)\ B(x) = (a_{ij}(x))\ (b_{ij}(x)) = [\sum_{k=1}^{n}\ a_{ik}(x)\ b_{kj}(x)]$.

So, for all $x \in J$, $\frac{d}{dx}\ [A(x)\ B(x)] = (\frac{d}{dx}\ \{\sum_{k=1}^{n}\ a_{ik}(x)\ b_{kj}(x)\})$

$$= (\sum_{k=1}^{n}\ \{a'_{ik}(x)\ b_{kj}(x) + a_{ik}(x)\ b'_{kj}(x)\})$$

$$= (\sum_{k=1}^{n}\ a'_{ik}(x)\ b_{kj}(x)) + (\sum_{k=1}^{n}\ a_{ik}(x)\ b'_{kj}(x))$$

Hence $\frac{d}{dx}\ [A(x)\ B(x)] = A'(x)\ B(x) + A(x)\ B'(x), \qquad (x \in J) \qquad (2.1)$

3. Derivative of $A^n(x)$, when $A'(x)\ A(x) = A(x)\ A'(x)\ (x \in J)$: (3.1)

Taking $B(x) = A(x)$ for all $x \in J$ in (2.1), one obtains, using (3.1),

$$\frac{d}{dx}\ A^2(x) = A'(x)\ A(x) + A(x)\ A'(x) = 2A'(x)\ A(x) = 2A(x)\ A'(x),\ x \in J. \qquad (3.2)$$

Assuming that, for some positive integer m,

$$\frac{d}{dx}\ A^m(x) = mA'(x)\ A^{m-1}(x) = mA^{m-1}(x)\ A'(x) \text{ holds for all } x \in J, \qquad (3.3)$$

one derives

$$\frac{d}{dx}\ A^{m+1}(x) = A'(x)\ A^m(x) + A(x)\ \frac{d}{dx}\ A^m(x), \qquad x \in J$$

$$= A'(x)\,A^m(x) + A(x).\ mA'(x)\,A^{m-1}(x), \qquad \text{[using (3.3)]}$$

$$= (m+1)\ A'(x)\ A^m(x),\ (x \in J). \qquad \text{[using (3.1)]}$$

Similarly it can be proved that

$$\frac{d}{dx}\ A^{m+1}(\mathrm{x}) = (m+1)\,A^m(x)\,A'(x), \quad (x \in J).$$

Hence, by the method of induction, it follows that, for any positive integer n,

$$\frac{d}{dx}\ A^n(x) = nA^{n-1}(x)\,A'(x) = nA'(x)\,A^{n-1}(x), \qquad (x \in J), \qquad (3.4)$$

$$\text{if } A'(x)\,A(x) = A(x)\,A'(x), \qquad (x \in J).$$

4. Derivative of $e^{A(x)}$ when $A'(x)A(x) = A(x)A'(x)$ $(x \in J)$:

It is noted that the series in (1.1) defining $e^{A(x)}$ is uniformly convergent on any interval J and so is its derived series. Hence the series (1.1) can be differentiated term by term.

So one obtains, for all $x \in J$

$$\frac{d}{dx}e^{A(x)} = \sum_{r=0}^{\infty} \frac{d}{dx}(\frac{1}{r!}A^{r}(x)) = \sum_{r=1}^{\infty} \frac{1}{(r-1)!}A^{r-1}(x)A'(x)$$

$$= \sum_{r=1}^{\infty} \frac{1}{(r-1)!}A'(x)A^{r-1}(x), \qquad \text{[using (3.4)]}$$

$$= \left\{\sum_{s=0}^{\infty} \frac{I}{s!}A^{s}(x)\right\}A'(x) = \left\{A'(x)\sum_{s=0}^{\infty} \frac{I}{s!}A^{s}(x)\right\}$$

i.e. $\frac{d}{dx}e^{A(x)} = e^{A(x)}.A'(x) = A'(x).e^{A(x)}, \qquad x \in J,$

provided $A'(x)\,A(x) = A(x)\,A'(x)$.

5. Derivative of $A^n(x)$ when $A(x), A'(x)$ may not commute :

If $A(x), A'(x)$ do not necessarily commute, it is clear from (3.2) that the derivative of $A^2(x)$ cannot be claimed to be $2A(x)\,A'(x)$ or $2A'(x)\,A(x)$.

In fact we can only derive from (2.1) that

$$\frac{d}{dx}A^2(x) = A'(x)\,A(x) + A(x)\,A'(x), \qquad x \in J.$$

Writing $C(A, B)$ for $A(x)\,B(x) - B(x)\,A(x), x \in J$, and omitting the variable x for brevity, we have

$$\frac{d}{dx}A^2(x) = \frac{d}{dx}A^2 = 2A'A + C(A, A') = 2AA' + C(A', A) \tag{5.1}$$

It is claimed that, for all positive integers n, and for all $x \in J$,

$$\frac{d}{dx}A^n(x) = \frac{d}{dx}A^n = nA^{n-1}A' + \sum_{k=1}^{n-1} A^{n-1-k}\;C(A', A^k), \tag{5.2}$$

$$= n\,A'A^{n-1} + \sum_{k=1}^{n-1} C(A^k, A')A^{n-1-k}. \tag{5.3}$$

(5.2) and (5.3) are proved below by the method of induction.

Assume that (5.2) holds for some positive integer m.

Then

$$\frac{d}{dx}A^{m+1} = \frac{d}{dx}(A^m.A) = (\frac{d}{dx}A^m)A + A^mA'$$

$$= \{mA^{m-1}A' + \sum_{k=1}^{m-1} A^{m-1-k}\ C(A', A^k)\}\ A + A^mA', \qquad \text{by (5.2)}$$

$$= A^m A' + A'A^m + AA'A^{m-1} + A^2A'A^{m-2} + \ldots + A^{m-2}A'A^2 + A^{m-1}A'A$$

$$= (m+1)\,A^mA' + (A'A^m - A^mA') + A(A'A^{m-1} - A^{m-1}A') + \ldots +$$

$$A^{m-2}(A'A^2 - A^2A') + A^{m-1}(A'A - AA')$$

$$= (m+1)\,A^mA' + \sum_{k=1}^{m-1} A^{m-k}\ C(A', A^k). \tag{5.4}$$

Hence, from (5.1) and (5.4) it follows by the method of induction that (5.2) holds for all positive integers n. Similarly it can be proved that (5.3) holds for all positive integers n.

6. Derivative of $e^{A(x)}$ when $A'(x), A(x)$ may not commute :

The facts that the infinite series (1.1) representing $e^{A(x)}$ is uniformly convergent on any finite interval J, and that its derived series is also uniformly convergent on J, ensure that the series (1.1) can be differentiated term by term.

So one derives

$$\frac{d}{dx}e^{A(x)} = \sum_{k=1}^{\infty}\frac{1}{n!}\frac{d}{dx}A^n$$

$$= A' + \sum_{n=2}^{\infty}\frac{1}{n!}\{nA^{n-1}A' + \sum_{k=1}^{n-1}A^{n-1-k}C(A', A^k)\}, \qquad \text{using (5.2)}$$

$$= \sum_{n=1}^{\infty}\frac{1}{(n-1)}A^{n-1}A' + \sum_{n=2}^{\infty}\frac{1}{n!}\sum_{k=1}^{n-1}A^{n-1-k}C(A', A^k)$$

$$= e^{A(x)}A'(x) + \sum_{n=0}^{\infty}\frac{1}{(n+2)!}\sum_{k=1}^{n+1}A^{n+1-k}C(A', A^k),$$ which proves (1.3).

Similarly (1.4) can be proved by using (5.3) instead of (5.2).

Note : It is to be noted that, if $A(x)$, $A'(x)$ commute,

$$C(A', A^k) = 0 = C(A^k, A') \qquad \text{for all positive integer, } k.$$

Hence if $A(x)$, $A'(x)$ commute,

$$\frac{d}{dx}e^{A(x)} = e^{A(x)}.\,A'(x) = A'(x).\,e^{A(x)}.$$

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EXAMPLES COMPARING LINEAR AND QUARDRATIC NATURAL BOUNDARY CONDITIONS ASSOCIATED WITH A SECOND ORDER LINEAR HOMOGENEOUS DIFFERENTIAL EQUATION

JAYASRI SETT AND SUSMITA DAS

ABSTRACT : We consider the second-order linear homogeneous Ordinary Differential Equation (ODE)

$$p_0(x)y''(x) + p_1(x)y'(x) + p_2(x)y(x) = 0 \tag{1}$$

where p_0, p_1, p_2 are continuous complex valued functions and $p_0(x) \neq 0$ for all real $x \in [a, b]$. We consider the Linear Boundary Condition (BC) of the form

$$U_\alpha[y] = \alpha_1 y(a) + \alpha_2 y'(a) + \alpha_3 y(b) + \alpha_4 y'^2(b) = 0 \tag{2}$$

and Quadratic Boundary Condition (QBC) of the form

$$V_\alpha[y] = \alpha_1 y^2(a) + \alpha_2 y'^2(a) + \alpha_3 y^2(b) + \alpha_4 y'^2(b) = 0 \tag{3}$$

where α_1, α_2, α_3, α_4 are real numbers, $U_\alpha[y]$ and $V_\alpha[y]$ are associated (1).

Let $\xi, \eta : [a, b) \rightarrow C$ denote the linearly independent solutions of (1) which satisfy the following initial conditions

$$\xi(a) = 1 \qquad \xi'(a) = 0 \tag{4}$$

$$\eta(a) = 0 \qquad \eta'(a) = 1 \tag{5}$$

Let

$$\xi(b) = \xi_1 + i\xi_2 \qquad \xi'(b) = \xi'_1 + i\xi'_2 \tag{6}$$

$$\eta(b) = \eta_1 + i\eta_2 \qquad \eta'(b) = \eta'_1 + i\eta'_2 \tag{7}$$

where ξ_1, ξ'_1, η_1, η'_1 $(i = 1, 2)$ are real numbers.

If all the solutions of (1) satisfy the BC (2) then the BC (2) is called the Natural Boundary Condition (NBC) corresponding to (1).

The necessary and sufficient condition for the DE (1) to possess a NBC of the form (2) is $\xi_2\eta'_2 - \xi'_2\eta_2 = 0$. This result and different cases of NBCs are established in the paper 'On the Boundary Conditions Associated with Second Order Linear Homogeneous Differential Equations' [Archivum Mathematicum Tomus 40 (2004)] by J. Das. [2]

If all solutions of the DE (1) satisfy QBC (3), then the QBC (3) is called Quadratic Natural Boundary Condition (QNBC) corresponding to (1). The necessary and sufficient condition for the DE (1) to possess a QNBC of the form (3) is that the coefficient determinant of any two equations of the following four equations

$$\left.\begin{aligned}
&\alpha_3(\xi_1\eta_1 - \xi_2\eta_2) + \alpha_4(\xi'_1\eta'_1 - \xi'_2\eta'_2) = 0\\
&\alpha_3(\xi_2\eta_1 - \xi_1\eta_2) + \alpha_4(\xi'_2\eta'_1 - \xi'_1\eta'_2) = 0\\
&\alpha_3\xi_1\xi_2 + \alpha_4\xi'_1\xi'_2 = 0\\
&\alpha_3\eta_1\eta_2 + \alpha_4\eta'_1\eta'_2 = 0
\end{aligned}\right\} \tag{A}$$

is zero. This result and different cases of QNBCs are established in the paper 'Quadratic Natural Boundary Conditions Associated with Second Order Linear Homogeneous Differential Equations' by Jayasri Sett and Susmita Das. This paper has been published in Bull. Cal. Math. Soc., 105(5), 325 – 332 (2013). [5]

In the present paper we consider examples examining whether for DE (1) linear NBCs and QNBCs (i) exist simultaneously, (ii) do not exist at all, (iii) NBC exists but QNBC does not exist, (iv) NBC does not exist but QNBC exists.

§1. INTRODUCTION

We consider the second-order linear homogeneous ODE

$$p_0(x)y''(x) + p_1(x)y'(x) + p_2(x)y(x) = 0 \tag{1.1}$$

where p_0, p_1, p_2 are continuous complex valued functions and $p_0(x) \neq 0$ for all real $x \in [a, b]$.

We consider two types of Boundary Conditions associated with (1.1).

I. Linear Boundary Condition (BC) of the form

$$U_\alpha[y] = \alpha_1 y(a) + \alpha_2 y'(a) + \alpha_3 y(b) + \alpha_4 y'(b) = 0 \tag{1.2}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are real numbers.

To explain the concept of Natural Boundary Condition (NBC) we consider the DE $y''(x) + y(x) = 0$ on $[0, \pi]$. The linearly independent solutions of this equation i.e., sin x and cos x satisfy the non-trivial linear homogeneous BCs $y(0) + y(\pi) = 0$ and $y'(0) + y'(\pi) = 0$. This means all solutions in the solution space of the given DE satisfy the BCs. Such BCs are referred to as Natural Boundary Conditions (NBC).

II. Quadratic Boundary Condition (QBC) of the form

$$V_\alpha[y] = \alpha_1 y^2(a) + \alpha_2 y'^2(a) + \alpha_3 y^2(b) + \alpha_4 y'^2(b) = 0 \tag{1.3}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are real numbers.

Here also we explain the concept of Quadratic Natural Boundary Condition (QNBC) by the example $y''(x) + y(x) = 0$. We note that all solutions of this DE satisfy the Quadratic BCs $y^2(0) - y^2(\pi) = 0$ and $y'^2(0) - y'^2(\pi) = 0$. Such BCs are referred to as Quadratic Natural Boundary Condition (QNBC).

Considering the above two examples the following questions may be asked

(i) whether there always exist a NBC and a QNBC with respect to the DE (1.1) or not.

(ii) If there exist NBC and QNBC, how many linearly independent NBCs and QNBCs are there with respect to the given DE? This paper is based on the following two papers which answers the above questions. One is 'On the Boundary Conditions Associated with Second-order Linear Homogenous Differential Equations' by J. Das [Archivum Mathematicum Tomus 40 (2004)] [2] and another is 'Quadratic Natural Boundary Conditions Associated with Second-order Linear Homogeneous Differential Equations' by Jayasri Sett and Susmita Das. [Bull. Cal. Math. Soc., 105(5), 325 – 332 (2013)]. [5]

The aim of the present paper is to consider examples comparing different combinations of existence and nonexistence of NBC and QNBC. There are nine cases to consider where a DE is given.

(i) NBC and QNBC both do not exist.

(ii) NBC does not exist but one QNBC exists.

(iii) NBC does not exist but two QNBCs exist.

(iv) One NBC exists but QNBC does not exist.

(v) One NBC and one QNBC exist.

(vi) One NBC exists and two QNBCs exist.

(vii) Two NBCs exist but QNBC does not exist.

(viii) Two NBCs exist but one QNBC exists.

(ix) Two NBCs and two QNBCs exist.

In §2 we have quoted from [2] the necessary and sufficient condition for the existence of NBC of type (1.2) and the number of NBCs for (1.1) in different cases. In §3 we have quoted from [5] the necessary and sufficient condition for the existence of QNBC of type (1.3) and the linearly independent QNBCs in different cases. §4 deals with the examples of cases (i) – (ix).

Before proceeding to §2 we give some necessary preliminary results.

Let $\xi, \eta : [a, b] \to C$ denote the linearly independent solutions of (1.1) that satisfy the following initial conditions

$$\xi(a) = 1 \qquad \xi'(a) = 0 \tag{1.4}$$

$$\eta(a) = 0 \qquad \eta'(a) = 1 \tag{1.5}$$

As the coefficients p_0, p_1, p_2 in (1.1) are complex valued the solutions ξ, η are complex valued.

Let $\xi(b) = \xi_1 + i\xi_2 \qquad \xi'(b) = \xi_1' + i\xi_2'$

$\eta(b) = \eta_1 + i\eta_2 \qquad \eta'(b) = \eta_1' + i\eta_2'$

where ξ_i, ξ_i', η_i, η_i' $(i = 1, 2)$ are real numbers.

§2. Theorem 1 : The necessary and sufficient condition for the DE (1.1) to possess a NBC of the form (1.2) is $\xi_2\eta_2' - \xi_2'\eta_2 = 0$.

Two cases have been considered

(i) $(\xi_2, \xi_2', \eta_2, \eta_2') \neq 0$

(ii) $(\xi_2, \xi_2', \eta_2, \eta_2') = 0$.

In case (i) the NBC with respect to the DE (1.1) is

$$(\xi_1\xi_2' - \xi_1'\xi_2)\, y(a) + (\xi_2'\eta_1 - \xi_2\eta_1')\, y'(a) - \xi_2' y(b) + \xi_2' y(b) = 0 \tag{2.1}$$

In case (ii) two linearly independent NBCs exist with respect to the DE (1.1) viz.,

$$\xi_1 y(a) + \eta_1 y'(a) - y(b) = 0 \tag{2.2}$$

$$\xi_1' y(a) + \eta_1' y'(a) - y'(b) = 0 \tag{2.3}$$

§3. Theorem 2 : The necessary and sufficient condition for the DE (1.1) to possess a QNBC of the form (1.3) is that the coefficient determinant of any two equations of the following four equations

$$\left.\begin{aligned} &\alpha_3(\xi_1\eta_1 - \xi_2\eta_2) + \alpha_4(\xi'_1\eta'_1 - \xi'_2\eta'_2) = 0 \\ &\alpha_3(\xi_2\eta_1 - \xi_1\eta_2) + \alpha_4(\xi'_2\eta'_1 - \xi'_1\eta'_2) = 0 \\ &\alpha_3\xi_1\xi_2 + \alpha_4\xi'_1\xi'_2 = 0 \\ &\alpha_3\eta_1\eta_2 + \alpha_4\eta'_1\eta'_2 = 0 \end{aligned}\right\} \tag{3.1}$$

is zero.

Two cases arise :

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Case (i) : If at least one element of the coefficient matrix of (3.1) is not zero (e.g., $\eta_1\eta_2 \neq 0$) then only one QNBC exists. The QNBC is of the form

$$[\eta'_1\eta'_2\xi_1{}^2 - \eta'_1\eta'_2\xi_2{}^2 - \eta_1\eta_2\xi'_1{}^2 + \eta_1\eta_2\xi'_2{}^2]\ y^2(a) +$$

$$[\eta'_1\eta'_2\eta_1{}^2 - \eta'_1\eta'_2\eta_2{}^2 - \eta_1\eta_2\eta'_1{}^2 + \eta_1\eta_2\eta'_2{}^2]\ y^2(a) - \eta'_1\eta'_2 y'^2(b) + \eta_1\eta_2 y'^2(b) = 0 \quad (3.2)$$

Case (ii) : If all the elements of the coefficient matrix of (3.1) are zero, then two linearly independent QNBCs exist.

They are of the form

$$(\xi_2{}^2 - \xi_1{}^2)\, y^2(a) + (\eta_2{}^2 - \eta_1{}^2)\, y^2(a) + y^2(b) = 0 \quad (3.3)$$

and $$(\xi'_2{}^2 - \xi'_1{}^2)\, y^2(a) + (\eta'_2{}^2 - \eta'_1{}^2)\, y'^2(a) + y'^2(b) = 0 \quad (3.4)$$

4. Examples of cases (i) – (ix)

Case (i) : Both NBC and QNBC do not exist.

Ex. Consider $y''(x) + 2iy'(x) - y(x) = 0, \qquad x \in [0, \pi]$ (4.1)

We have $\xi(x) = (1 + ix)e^{-ix}$ and $\eta(x) = x\ e^{-x}$

$\xi'(x) = x\ e^{ix}$ and $\eta'(x) = e^{-ix} - ix\ e^{-ix}$

$\therefore\ \xi_1 = -1;\ \xi_2 = -\pi,\ \xi'_1 = -\pi,\ \xi'_2 = 0$

$\eta_1 = -\pi,\ \eta_2 = 0,\ \eta'_1 = -1,\ \eta'_2 = \pi.$

Then $\xi_2\eta'_2 - \xi'_2\eta_2 = -\pi^2 \neq 0$

So NBC does not exist by Theorem 1 in §2.

Also after calcutation the coefficient determinant of any two equations of (3.1) is not zero. So QNBC does not exist by Theorem 2 in §3.

Case (ii) NBC does not exist but one QNBC exists

Ex. Consider $y''(x) + iy'(x) + 6y(x) = 0, \qquad x \in [0, \pi]$ (4.2)

We have $\xi(x) = \frac{2}{5}e^{-3ix} + \frac{3}{5}e^{2ix}$

and $\eta(x) = \frac{i}{5}e^{-3ix} - \frac{i}{5}e^{2ix}$

$\xi'(x) = \frac{-6i}{5}e^{-3ix} + \frac{6i}{5}e^{2ix}$

$\eta'(x) = \frac{3}{5}e^{-3ix} + \frac{2}{5}e^{2ix}$

Here $\xi_1 = \frac{1}{5}$, $\xi_2 = 0$, $\xi'_1 = 0$, $\xi'_2 = \frac{12}{5}$

$\eta_1 = 0$, $\eta_2 = -\frac{2}{5}$ $\eta'_1 = -\frac{1}{5}$ $\eta'_2 = 0$

Here $\xi_2\eta'_2 - \xi'_2\eta_2 \neq 0$.

So NBC does not exist by Theorem 1 of §2.

Here $\xi'_2\eta'_1 + \xi'_1\eta'_2 = -\frac{12}{25} \neq 0$ i.e., one element of the coefficient matrix of (3.1) is not zero. The coefficient determinant of any two equations of (3.1) is zero. Hence one QNBC exists by case (i) of Theorem 2 of §3.

This QNBC is of the form

$$[(\xi'_2\eta'_1 + \xi'_1\eta'_2)\xi'^2_1 - (\xi'_2\eta'_1 + \xi'_1\eta'_2)\xi^2_2 - \xi'^2_1(\xi_2\eta_1 + \xi_1\eta_2) + \xi'^2_2(\xi_2\eta_1 + \xi_1\eta_2)]y^2(a)$$
$$+ [(\xi'_2\eta'_1 + \xi'_1\eta'_2)\eta^2_1 - (\xi'_2\eta'_1 + \xi'_1\eta'_2)\eta^2_2 - \eta'^2_1(\xi_2\eta_1 + \xi_1\eta_2) + \eta'^2_2(\xi_2\eta_1 + \xi_1\eta_2)]y'^2(a)$$
$$- (\xi'_2\eta'_1 + \xi'_1\eta'_2)\, y^2(b) + (\xi_2\eta_1 + \xi_1\eta_2)y'^2(b) = 0$$

Substituting the values the QNBC becomes

$$6(y^2(0) - y^2(\pi)) - (y'^2(0) - y'^2(\pi)) = 0 \tag{4.3}$$

Case (iii) NBC does not exist but two QNBCs exists.

Consider $y''(x) + 2iy'(x) + 3y(x) = 0, \qquad x \in \left[0, \frac{\pi}{2}\right] \qquad (4.4)$

We have $\xi(x) = \dfrac{e^{-3ix} + 3e^{ix}}{4}$

and $\eta(x) = \dfrac{2ie^{ix} - 2ie^{-3ix}}{-8}$

$$\xi'(x) = \frac{-3ie^{-3ix} + 3ie^{ix}}{4}$$

$$\eta'(x) = \frac{-2e^{ix} - 6e^{-3ix}}{-8}$$

Here $\xi_1 = 0, \quad \xi_2 = 1, \quad \eta_1 = 0, \quad \eta_2 = 0$

$\xi'_1 = 0, \quad \xi'_2 = 0, \quad \eta'_1 = 0, \quad \eta'_2 = 1$

Here $\xi_2\eta'_2 - \xi'_2\eta_2 \neq 0.$

Hence NBC does not exist by Theorem 1 of §2.

But all elements of the coefficient matrix of (3.1) are zero. Hence two QNBCs of the form (3.3) and (3.4) exist by case (ii) of Theorem 2 of §3.

These QNBCs are $y^2(0) + y^2\left(\frac{\pi}{2}\right) = 0 \qquad (4.5)$

$$y'^2(0) + y'^2\left(\frac{\pi}{2}\right) = 0 \qquad (4.6)$$

Case (iv) : One NBC exists but QNBC does not exists.

Consider $(1 - ix^2)\, y''(x) + 2ixy'(x) - 2iy(x) = 0, \quad x \in [0, \pi]$ (4.7)

Here $\xi(x) = 1 + ix^2$ and $\eta(x) = x$

$\xi'(x) = 2ix$ and $\eta'(x) = 1.$

Here $\xi(\pi) = 1 + i\pi^2 \Rightarrow \xi_1 = 1, \quad \xi_2 = \pi^2$

$\xi'(\pi) = 2i\pi \Rightarrow \xi'_1 = 0, \quad \xi'_2 = 2\pi$

$\eta(\pi) = \pi \Rightarrow \eta_1 = \pi, \quad \eta_2 = 0$

$\eta'(\pi) = 1 \Rightarrow \eta'_1 = 1, \quad \eta'_2 = 0$

Here $\xi_2\eta'_2 - \xi'_2\eta_2 = 0$ and $\xi'_2 \neq 0 \quad \xi_2 = 0$

One NBC exists by case (i) of Theorem 1 of §2.

This is $2(y(0) - y(\pi)) + \pi(y'(0) + y'(\pi) = 0.$ (4.8)

After calculation the coefficient determinant of any two equations of (3.1) are not zero. So QNBC does not exist by Theorem 3 of §3.

Case v : One NBC and one QNBC exists.

Consider $y''(x) - iy'(x) = 0, \quad x \in [0, \pi]$ (4.9)

We have $\xi(x) = 1$ and $\eta(x) = i(1 - e^{ix})$

$\xi(\pi) = 1 \Rightarrow \xi_1 = 1, \quad \xi_2 = 0$

$\xi'(\pi) = 0 \Rightarrow \xi'_1 = 0, \quad \xi'_2 = 0$

$\eta(\pi) = 2i \Rightarrow \eta_1 = 0, \quad \eta_2 = 2$

$\eta'(\pi) = -1 \Rightarrow \eta'_1 = -1, \quad \eta'_2 = 0.$

Here $\xi_2\eta'_2 - \xi'_2\eta_2 = 0$ but $\eta'_2 \neq 0$

Hence one NBC exists by case (i) of Theorem 1 in §2.

This NBC is $y'(0) + y'(\pi) = 0$ (4.10)

Also one element $\xi_1\eta_2 + \xi_2\eta_1 = 2$ of the coefficient matrix of (3.1) is not zero. But the coefficient determinant of any two equations of (3.1) is zero.

Hence by case (i) of §3 of Theorem 2, one QNBC exists.

After calculation this QNBC is $y'^2(0) - y'^2(\pi) = 0$ (4.11)

Case (vi) : One NBC exists but two QNBCs exists.

Consider $(3e^{-3ix} + e^{ix})\, y''(x) - (-9ie^{-3ix} + ie^{ix})\, y'(x) = 0,$ $x \in \left[0, \frac{\pi}{2}\right]$ (4.12)

Hence $\xi(x) = 1$ and $\eta(x) = \dfrac{ie^{-3ix} - ie^{ix}}{4}$

$\xi'(x) = 0$ and $\eta'(x) = \dfrac{3e^{-3ix} + e^{ix}}{4}$

$$\xi\left(\frac{\pi}{2}\right) = 1 \Rightarrow \xi_1 = 1,\ \xi_2 = 0$$

$$\xi'\left(\frac{\pi}{2}\right) = 0 \Rightarrow \xi_1' = 0,\ \xi_2' = 0$$

$$\eta\left(\frac{\pi}{2}\right) = 0 \Rightarrow \eta_1 = 0,\ \eta_2 = 0$$

$$\eta'\left(\frac{\pi}{2}\right) = i \Rightarrow \eta_1' = 0,\ \eta_2' = 1.$$

Here $\xi_2\eta'_2 - \xi'_2\eta_2 = 0$ but $\eta'_2 \neq 0$

Hence one NBC exists by case (i) of Theorem 1 of §2.

This is $y(0) - y\left(\frac{\pi}{2}\right) = 0.$ (4.13)

All elements of the coefficient matrix of (3.1) are zero. Hence two QNBCs exist by case (ii) of Theorem 2 of §3.

These QNBCs are $y^2(0) - y^2\left(\frac{\pi}{2}\right) = 0$ (4.14)

and $y'^2(0) + y'^2\left(\frac{\pi}{2}\right) = 0$ (4.15)

Case (vii) : Two NBCs exist but one QNBC does not exist.

This case is not possible.

If two NBCs exist then $\xi_2\eta'_2 - \xi'_2\eta_2 = 0$ and also $\xi_2 = \xi'_2 = \eta_2 = \eta'_2 = 0$.

Then the coefficient determinant of any two equations of (3.1) is zero so at least one QNBC will always exist.

Case (viii) : Two NBCs exist and one QNBC exist.

Here $xy''(x) + 2y'(x) + \frac{1}{4}xy(x) = 0, \qquad x \in [\pi, 2\pi]$ (4.16)

Similar examples are given in [1 lecture 30].

Here
$$\xi(x) = \frac{\pi \sin\frac{x}{2}}{x} - \frac{2\cos\frac{x}{2}}{x}$$

$$\eta(x) = -\frac{2\pi\cos\frac{x}{2}}{x}$$

$$\xi'(x) = -\frac{1}{x^2}\pi\sin\frac{x}{2} + \frac{\pi}{x}.\frac{1}{2}\cos\frac{x}{2} + \frac{1}{x^2}.2\cos\frac{x}{2} + \frac{\sin\frac{x}{2}}{x}$$

$$\eta'(x) = \frac{2\pi}{x^2}\cos\frac{x}{2} + \frac{\pi\sin\frac{x}{2}}{x}$$

$$\xi(2\pi) = \frac{1}{\pi} \Rightarrow \xi_1 = \frac{1}{\pi}, \xi_2 = 0$$

$$\xi'(2\pi) = -\frac{1}{2\pi^2} - \frac{1}{4} \Rightarrow \xi'_1 = -\frac{1}{2\pi^2} - \frac{1}{4}, \; \xi'_2 = 0.$$

$\eta(2\pi) = 1 \Rightarrow \eta_1 = 1, \quad \eta_2 = 0.$

$\eta'(2\pi) = -\frac{1}{2\pi} \Rightarrow \eta_1 = -\frac{1}{2\pi}, \; \eta'_2 = 0.$

Hence $\quad \xi_2\eta'_2 - \xi'_2\eta_2 = 0$ and $\xi_2 = \xi'_2 = \eta_2 = \eta'_2 = 0.$

Hence two NBCs exists by case (ii) of Theorem 1 of §2.

These QNBC are $\quad y(\pi) + \pi y'(\pi) - \pi y(2\pi) = 0. \qquad (4.17)$

$$(\pi^2 + 2)\, y(\pi) + 2\pi y'(\pi) + 4\pi^2 y'(2\pi) = 0 \qquad (4.18)$$

Again $\xi_1\eta_1 - \xi_2\eta_2 = \frac{1}{\pi} \neq 0$ and $\xi'_1\eta'_1 - \xi'_2\eta'_2 = \frac{\pi^2+2}{8\pi^3} \neq 0$. The coefficient determinant of any two equations of (3.1) is zero.

Hence by case (i) of Theorem 2 of §3 one QNBC exists.

This is given by

$$[(\xi'_1\eta'_1 - \xi'_2\eta'_2)\,\xi^2_1 - (\xi'_1\eta'_1 - \xi'_2\eta'_2)\,\xi^2_2 - (\xi_1\eta_1 - \xi_2\eta_2)\,\xi'^2_1 + (\xi_1\eta_1 - \xi_2\eta_2)\xi'^2_2]y^2\,(a)$$
$$+ (\xi'_1\eta'_1 - \xi'_2\eta'_2)\,\eta^2_1 - (\xi'_1\eta'_1 - \xi'_2\eta'_2)\,\eta^2_2 - (\xi_1\eta_1 - \xi_2\eta_2)\eta'^2_1 + (\xi_1\eta_1 - \xi_2\eta_2)\eta'^2_2\,]y'^2\,(a)$$
$$- (\xi'_1\eta'_1 - \xi'_2\eta'_2)\; y^2\,(b) + (\xi_1\eta_1 - \xi_2\eta_2)\; y'^2\,(b) = 0$$

Substituting the values the QNBC is

$$(\pi^2 + 2)y^2\,(\pi) - 2\pi^2 y'^2(\pi) + 2(\pi^2 + 2)y^2\,(2\pi) - 16\pi^2 y'^2(2\pi) = 0 \qquad (4.19)$$

Case ix : Two NBCs and two QNBCs exist.

Ex. $\quad y''(x) + y(x) = 0. \quad x \in [0, \pi] \qquad (4.20)$

We have $\quad \xi(x) = \cos x, \qquad \eta(x) = \sin x.$

$\xi'(x) = -\sin x, \qquad \eta'(x) = \cos x.$

$\xi(\pi) = -1 \qquad \Rightarrow \quad \xi_1 = -1, \qquad \xi_2 = 0$

$$\xi'(\pi) = 0 \quad \Rightarrow \quad \xi'_1 = 0, \quad \xi'_2 = 0$$

$$\eta(\pi) = 0 \quad \Rightarrow \quad \eta_1 = 0, \quad \eta_2 = 0$$

$$\eta'(\pi) = -1 \quad \Rightarrow \quad \eta'_1 = -1, \quad \eta'_2 = 0.$$

Here $\quad \xi_2\eta'_2 - \xi'_2\eta_2 = 0$ and $\xi_2 = \xi'_2 = \eta_2 = \eta'_2 = 0.$

Hence one NBCs exists by case (ii) of Theorem 1 of §2.

They are $\quad y(0) + y(\pi) = 0 \qquad (4.21)$

$$y'(0) + y'(\pi) = 0 \qquad (4.22)$$

Also all elements of the coefficient matrix of (3.1) are zero. Hence two QNBCs exist by case (ii) of Theorem 2 of §3.

They are $\quad y^2(0) - y^2(\pi) = 0 \qquad (4.23)$

$$y'^2(0) - y'^2(\pi) = 0 \qquad (4.24)$$

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GENERALIZED β CONNECTEDNESS IN INTUITIONISTIC FUZZY TOPOLOGLCAL SPACES

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ABSTRACT : In this paper we have introduced intuitionistic fuzzy generalized β connected space, intuitionistic fuzzy generalized β super connected space and the intuitionistic fuzzy generalized β extremally disconnected space. We investigated some of their properties. Also we characterized the intuitionistic fuzzy generalized β super connected space.

Key words and phrases : Intuitionistic fuzzy topology, intuitionistic fuzzy generalized β connected space and intuitionistic fuzzy generalized β super connected space.

1. INTRODUCTION

Zadeh [11] introduced the notion of fuzzy sets. After that there have been a number of generalizations of this fundamental concept. Atanassov [1] introduced the notion of intuitionistic fuzzy sets. Using the notion of intuitionistic fuzzy sets, Coker [3] introduced the notion of intuitionistic fuzzy topological space. Connectedness in intuitionistic fuzzy special topological spaces was introduced by Selma Oscag and Dogan Coker [7]. In this paper we have introduced intuitionistic fuzzy generalized β connected space, intuitionistic fuzzy generalized β super connected space and the intuitionistic fuzzy generalized β extremally disconnected space. We investigated some of their properties. Also we characterized the intuitionistic fuzzy generalized β super connected space.

2. PRELIMINARIES

Definition 2.1 : [1] *An intuitionistic fuzzy set* (IFS in short) A in X is an object having the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \ / \ x \in X\}$$

where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership

(nemaly $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) \leq 1$ for each $x \in X$. Denote by IFS (X), the set of all intuitionistic fuzzy sets in X.

Definition 2.2 : [1] Let A and B be IFSs of the form $A = \{<x, \mu_A(x), \nu_A(x)> / x \in X\}$ and $B = \{<x, \mu_B(x), \nu_B(x)> / x \in X\}$. Then

(a) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$

(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

(c) $A_c = \{<x, \nu_A(x), \mu_A(x)> / x \in X\}$

(d) $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)> / x \in X\}$

(e) $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)> / x \in X\}$

The intuitionistic fuzzy sets $0\sim = \{<x, 0, 1> / x \in X\}$ and $1\sim = \{<x, 1, 0> / x \in X\}$ are respectively they empty set and the whole set of X.

For the sake of simplicity, we shall use the notation $A = <x, \mu_A, \nu_A>$ istead of $A = \{<x, \mu_A(x), \nu_A(x)> / x \in X\}$.

Definition 2.3 : [3] An *intuitionistic fuzzy topology* (IFT for short) on X is a family τ of IFSs in X satisfying the following axioms.

(i) $0\sim, 1\sim \in \tau$

(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$

(iii) $\cup G_i \in \tau$ for any family $\{G_i / i \in J\} \subseteq \tau$.

In this case the pair (X, τ) is called in *intuitionistic fuzzy topological space* (IFTS in short) and any IFS in τ is known as an intuitionistic fuzzy open set (IFOS in short) in X. The complement A^c of an IFOS A in IFTS (X, τ) is called an intuitionistic fuzzy closed set (IFCS in short) in X.

Definition 2.4 : [10] An IFS $A = <x, \mu_A, \nu_A>$ in an IFTS (X, τ) is said to be

(i) *intuitionistic fuzzy β closed set* (IFβCS for short) if $int(cl(int(A))) \subseteq A$.

(ii) *intuitionistic fuzzy β open set* (IFβOS for short) if $A \subseteq cl(int(cl(A)))$.

Definition 2.5 : [4] Let A be an IFS in an IFTS (X, τ). Then the β-interior and the β-closure of A are defined as

$\beta int\,(A) = \cup\,\{G \,/\, G$ is an IFβOS in X and $G \subseteq A\}$.

$\beta cl\,(A) = \cap\,\{K \,/\, K$ is an IFβCS in X and $A \subseteq K\}$.

Note that for any IFS A in (X, τ), we have $\beta cl(A^c) = (\beta int(A))^c$ and $\beta int(A^c) = (\beta cl(A))^c$ [4].

Definition 2.6 : [8] An IFS A is an *intuitionistic fuzzy generalized closed set* (IFGCS for short) if $cl(A) \subseteq U$ wherever $A \subseteq U$ and U is an IFOS. The complement of an IFGCS is called an *intuitionistic fuzzy generalized open set* (IFGOS for short).

Definition 2.7 : [8] Two IFSs A and B are said to be *q-coincident* (A q B in short) if and only if there exists an element $x \in X$ such that $\mu_A(x) > \nu_B(x)$ or $\nu_A(x) < \mu_B(x)$.

Definition 2.8 : [8] Two IFSs A and B are said to be *not q-coincident* (A q^c B in short) if and only if $A \subseteq B_c$.

Definition 2.9 : [8] An IFTS (X, τ) is said to be an *$IFT_{1/2}$ space* if every IFGCS in (X, τ) is an IFCS in (X, τ).

Definition 2.10 : [4] An IFS A in the IFTS (X, τ) is said to be an *intuitionistic fuzzy generalized β closed set* (IFGβCS for short) if $\beta cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an IFOS in (X, τ). Every IFCS, IFβCS is an IFGβCS but the separate converses may be true in general. The complement A^c of an IFGβCS A in an IFTS (X, τ) is called an *intuitionistic fuzzy generalized β open set* (IFGβOS for short) in X. Every IFOS, IFβOS is an IFGβOS but the separate converses may not be true in general.

Definition 2.11 : [4] Let A be an IFS in an IFTS (X, τ). Then the generalized β-interior and the generalized β-closure of A are defined as

$g\beta int\,(A) = \cup\,\{G \,/\, G$ is an IFβOS in X and $G \subseteq A\}$.

$g\beta cl\,(A) = \cap\,\{K \,/\, K$ is an IFβCS in X and $A \subseteq K\}$

Note that for any IFS A in (X, τ), we have $g\beta cl(A^c) = (g\beta int(A))^c$ and $g\beta int(A^c) = (g\beta cl(A))^c$.

Definition 2.12 : [5] A mapping $f : (X, \tau) \to (Y, \sigma)$ is called an *intuitionistic fuzzy generalized β continuous* (IFGβ continuous for short) mapping if $f^{-1}(V)$ is an IFGβCS in (X, τ) for every IFCS V of (Y, σ).

Definition 2.13 : [5] A mapping $f : (X, \tau) \to (Y, \sigma)$ is called *intuitionistic fuzzy generalized β irresolute* (IFGβ irresolute) mapping if $f^{-1}(V)$ is an IFGβCS in (X, τ) for every IFGβCS V of (Y, σ).

Definition 2.14 : [4] If every IFGβCS in (X, τ) is an IFβCS in (X, τ), then the space can be called as an *intuitionistic fuzzy β $T_{1/2}$ space* (IFβ$T_{1/2}$ space for short).

Definition 2.15 : [4] An IFTS (X, τ) is said to be an intuitionistic fuzzy β $T^*_{1/2}$ space if every IFGβCS is an IFCS in (X, τ).

Definition 2.16 : [9] An IFTS (X, τ) is said to be ain intuitionistic fuzzy C_5-connected (IFC_5-connected for short) space if the only IFSs which are both intuitionistic fuzzy open and intuitionistic fuzzy closed are 0~ and 1~.

Definition 2.11 : [8] An IFTS (X, τ) is said to be ain intuitionistic fuzzy GO-connected (IFCO-connected for short) space if the only IFSs which are both intuitionistic fuzzy generalized open and intuitionistic fuzzy generalized closed are 0~ and 1~.

Definition 2.12 : [6] An IFS (X, τ) is intuitionistic fuzzy C_5-connected between two IFSs A and B if there is no intuitionistic fuzzy open set E in (X, τ) such that $A \subsetneq E$ and $E\ q^c B$.

3. INTUITIONISTIC FUZZY GENERALIZED β CONNECTED SPACES

In this section we introduce intuitionistic fuzzy generalized β connected space and intuitionistic fuzzy generalized β super connected space. We investigate some of their properties.

Also we provide a characterization theorem for an intuitionistic fuzzy generalized β super connected space.

Definition 3.1 : An IFTS (X, τ) is said to be an intuitionistic fuzzy generalized β connected space if the only IFSs which are both intuitionistic fuzzy generalized β open and intuitionistic fuzzy generalized β closed are 0~ and 1~.

Example 3.2 : Let X = {a, b} and τ = {0~, M, 1~} be an IFT on X, where M = <x, (0.5_a, 0.6_b), (0.5_a, 0.4_b)>. Then (X, τ) is an intuitionistic fuzzy generalized β connected space.

Theorem 3.3 : Every intuitionistic fuzzy generalized β connected space is an intuitionistic fuzzy C_5-connected space but not conversely.

Proof : Let (X, τ) be an intuitionistic fuzzy generalized β connected space. Suppose (X, τ) is not an intuitionistic fuzzy C_5-connected space, then there exists a proper IFS A which is both intuitionistic fruzzy open and intuitionistic fuzzy closed in (X, τ). That is A is both intuitionistic fuzzy generalized β open and intuitionistic fuzzy generalized β closed in (X, τ). This implies that (X, τ) is not an intuitionistic fuzzy generalized β connected space. This is a contradiction. Therefore (X, τ) must be an intuitionistic fuzzy C_5-connected space.

Example 3.4 : Let X = {a, b} and τ = {0~, M, 1~} be an IFT on X, where M = <x, (0.5_a, 0.4_b), (0.5_a, 0.6_b)>. Then (X, τ) is an intuitionistic fuzzy C_5-connected space but not an intuitionistic fuzzy generalized β connected space, since the IFS M in τ is both intuitionistic fuzzy generalized β open and intuitionistic fuzzy generalized β closed set in (X, τ).

Theorem 3.5 : Every intuitionistic fuzzy generalized β connected space is an intuitionistic fuzzy GO-connected space but not conversely.

Proof : Let (X, τ) be an intuitionistic fuzzy generalized β connected space. Suppose (X, τ) is not an intuitionistic fuzzy GO-connected space, then there exists a proper IFS A which is both intuitionistic fruzzy g-open and intuitionistic fuzzy g-closed in (X, τ). That is A is both intuitionistic fuzzy generalized β open and intuitionistic fuzzy generalized β closed in (X, τ). This implies that (X, τ) is not an intuitionistic fuzzy generalized β connected space. This is a contradiction. Therefore (X, τ) must be an intuitionistic fuzzy GO-connected space.

Example 3.6 : Let X = {a, b} and G = <x, (0.5, 0.6), (0.5, 0.4)>. Then τ = {0~, G, 1~} is an IFT on X. Let A = <x, (0.5, 0.7), (0.5, 0.3)> be an IFS in X. Then (X, τ) is an intuitionistic fuzzy GO-connected space but not an intuitionistic fuzzy generalized β connected space.

Theorem 3.7 : The IFTS (X, τ) is an intuitionistic fuzzy generalized β connected space if

and only if there exists no non-zero intuitiionistic fuzzy generalized β open set A and B in (X, τ) such that $A = B^c$.

Proof : Necessity : Let A and B be two intuitionistic fuzzy generalized β open sets in (X, τ) such that $A \neq 0\sim \neq B$ and $A = B^c$. Therefore B^c is an intuitionistic fuzzy generalized β closed set. Since $A \neq 0\sim$, $B \neq 1\sim$. This implies B is a proper IFS which is both intuitionistic fuzzy generalized β open and intuitionistic fuzzy genealized β closed in (X, τ). Hence (X, τ) is not an intuitionistic fuzzy generalized β connected space. But this is a contradiction to our hypothesis. Thus there exists no non-zero intuitionistic fuzzy generalized β open sets A and B in (X, τ) such that $A = B^c$.

Sufficiency : Let A be both intuitionstic fuzzy generalized β open and intuitionistic fuzzy generalized β closed in (X, τ) such that $1\sim \neq A \neq 0\sim$. Now let $B = A^c$. Then B is an intuitionistic fuzzy generalized β open and $B \neq 1\sim$. This implies $B = A^c \neq 0\sim$, which is a contradiction to our hypothesis. Therefore (X, τ) is an intuitionistic fuzzy generalized β connected space.

Theorem 3.8 : An IFTS (X, τ) is an intuitionistic fuzzy generalized β connected space if and only if there exists no non-zero intuitionistic fuzzy generalized β open sets A and B in (X, τ) such that $B = A^c$, $B = (\beta cl(A))^c$, $A = (\beta cl(B))^c$.

Proof : Necessity : Assume that there exists IFSs A and B such that $A \neq 0\sim \neq B$, $B = A^c$, $B = (\beta cl(A))^c$, $A = (\beta cl(B))^c$. Since $(\beta cl(A))^c$ and $(\beta cl(A))^c$ and $(\beta cl(B))^c$ are intuitionistic fuzzy generalized β open sets in (X, τ), A and B are intuitionistic fuzzy generalized β open sets in (X, τ). This implies (X, τ) is not an intuitionistic fuzzy generalized β connected space, which is a contradiction. Therefore there exists no non-zero intuitionistic fuzzy generalized β open sets A and B in (X, τ) such that $B = A^c$, $B = (\beta cl(A))^c$, $A = (\beta cl(B))^c$.

Sufficiency : Let A be both intuitionstic fuzzy generalized β open and intuitionistic fuzzy generalized β closed in (X, τ) such that $1\sim \neq A \neq 0\sim$. Now by taking $B = A^c$ we obtain a contradiction to our hypothesis. Therefore (X, τ) is an intuitionistic fuzzy generalized β connected space.

Theorem 3.9 : Let (X, τ) be an intuitionistic fuzzy β $T^*_{1/2}$ space, then the following are equivalent.

(i) (X, τ) is an intuitionistic fuzzy generalized β connected space.

(ii) (X, τ) is an intuitionistic fuzzy GO-connected space.

(iii) (X, τ) is an intuitionistic fuzzy C_5-connected space.

Proof : (i) ⇒ (ii) is obvious from the Theorem 3.5.

(ii) ⇒ (iii) is obvious from [7].

(iii) ⇒ (i) Let (X, τ) be an intuitionistic fuzzy C_5 connected space. Suppose (X, τ) is not an intuitionistic fuzzy generalized β coonected space, then there exists a proper IFS A in (X, τ) which is both intuitionistic fuzzy generalized β open and intuitionistic fuzzy generalized β closed in (X, τ). But since (X, τ) is an intuitionistic fuzzy β $T^*_{1/2}$ space, A is both intuitionistic fuzzy open and intuitionistic fuzzy closed in (X, τ). This implies that (X, τ) is not an intuitionistic fuzzy C_5 connected, which is a contradiction to our hypothesis. Therefore (X, τ) must be an intuitionistic fuzzy generalized β connected space.

Theorem 3.10 : If f : (X, τ) → (Y, σ) is an intuitionistic fuzzy generalized β continuous surjection and (X, τ) is an intitionistic fuzzy generalized β connected space, then (Y, σ) is an intuitionistic fuzzy C_5 connected space.

Proof : Let (X, τ) be an intuitionistic fuzzy generalized β connected space. Suppose (Y, σ) is not an intuitionistic fuzzy C_5 connected space, then there exists a proper IFS A which is both intuitionistic fuzzy open and intuitionistic fuzzy closed in (Y, σ). Since f is an intuitionistic fuzzy generalized β continous mapping, f^{-1} (A) is both intuitionistic fuzzy generalized β open and intuitionistic fuzzy generalized β closed in (X, τ). But this is a contradition to our hypothesis. Hence (Y, σ) must be an intuitionistic fuzzy C_5 connected space.

Theorem 3.11 : If f : (X, τ) → (Y, σ) is an intuitionistic fuzzy generalized β irresolute surjection and (X, τ) is an intuitionistic fuzzy generalized β connected space, then (Y, σ) is also an intuitionistic fuzzy generalized β connected space.

Proof : Suppose (Y, σ) is not an intuitionistic fuzzy generalized β connected space, then there exists a proper IFS A such that A is both intuitionistic fuzzy generalized β open and intuitionistic fuzzy generalized β closed in (Y, σ). Since f is an intuitionistic fuzzy generalized β irresolute

mapping, f^{-1} (A) is both intuitionistic fuzzy generalized β open and intuitionistic fuzzy generalized β closed in (X, τ). But this is a contradition to our hypothesis. Hence (Y, σ) must be an intuitionistic fuzzy generalized β connected space.

Definition 3.12 : An IFS (X, τ) is intuitionistic fuzzy generalized β connected between two IFSs A and B if there is no intuitionistic fuzzy genralized β open set E in (X, τ) such that $A \subseteq E$ and $E\ q^c\ B$.

Example 3.13 : Let $X = \{a, b\}$ and $\tau = \{0_\sim, M, 1_\sim\}$ be an IFT on X, where M = <x, $(0.5_a, 0.3_b)$, $(0.5_a, 0.1_b)$>. The (X, τ) is intuitionistic fuzzy generalized β connected between the two IFSs A = <x, $(0.5_a, 0.4_b)$, $(0.5_a, 0.3_b)$> and B = <x, $(0.5_a, 0.2_b)$, $(0.5_a, 0.5_b)$>.

Theorem 3.14 : If an IFTS (X, τ) is intuitionistic fuzzy generalized β connected between two IFSs A and B, then it is intuitionistic fuzzy C_5-connected between A and B but the converse may not be true in general.

Proof : Suppose (X, τ) is not intuitionistic fuzzy C_5-connected between A and B, then there exists an intuitionistic fuzzy open set E in (X, τ) such that $A \subseteq E$ and $E\ q^c\ B$. Since every intuitionistic fuzzy open set is an intuitionistic fuzzy generalized β open set, there exists an intuitionistic fuzzy generalized β open set E in (X, τ) such that $A \subseteq E$ and $E\ q^c B$. This implies (X, τ) is not intuitionistic fuzzy generalized β connected between A and B, a contradiction to our hypothesis. Therefore (X, τ) must be intuitionistic fuzzy C_5-connected between A and B.

Example 3.15 : Let $X = \{a, b\}$ and $\tau = \{0_\sim, M, 1_\sim\}$ be an IFT on X, where M = <x, $(0.4_a, 0.3_b)$, $(0.2_a, 0.3_b)$>, then (X, τ) is intuitionistic fuzzy C_5-connected between the IFSs A = <x, $(0.4_a, 0.4_b)$, $(0.6_a, 0.6_b)$> and B = <x, $(0.5_a, 0.4_b)$, $(0.5_a, 0.6_b)$>. But (X, τ) is not an intuitionistic fuzzy generalized β connected between A and B, since the IFS E = (x, $(0.4_a, 0.4_b)$, $(0.5_a, 0.5_b)$> is an intuitionistic fuzzy generalized β open set such that $A \subseteq E$ and $E \subseteq B^c$.

Theorem 3.16 : An IFTS (X, τ) is intuitionistic fuzzy generalized β connected between two IFSs A and B if and only if there is no intuitionistic fuzzy generalized β open set and intuitionistic fuzzy generalized β closed set E in (X, τ) such that $A \subseteq E$ and $E \subseteq B^c$.

Proof : Necessity : Let (X, τ) is not intuitionistic fuzzy generalized β between A and B. Suppose that there exists an intuitionistic fuzzy generalized β open set and intuitionistic fuzzy generalized β closed set E in (X, τ) such that $A \subseteq E \subseteq B^c$, then $E\ q^c\ B$ and $A \subseteq E$. This implies (X, τ) is not intuitionistic fuzzy generalized β connected between A and B, by Definition 3.12. A contradiction to our hypothesis. Therefore there exists no intuitionistic fuzzy generalized β open set and intuitionistic fuzzy generalized β closed set E in (X, τ) such that $A \subseteq E \subseteq B^c$.

Sufficiency : Suppose that (X, τ) is not intuitionistic fuzzy generalized β connected between A and B. Then there exists an intuitionistic fuzzy generalized β open set E in (X, τ) such that $A \subseteq E$ and $E\ q^c\ B$. This implies that there exists an intuitionistic fuzzy generalized β open set E in (X, τ) such $A \subseteq E \subseteq B_c$. But this is a contradiction to our hypothesis. Hence (X, τ) must be intuitionistic fuzzy generalized β connected between A and B.

Theorem 3.17 : If an IFTS (X, τ) is intuitionistic fuzzy generalized β connected between two A and B and $A \subseteq A_1$, $B \subseteq B_1$, then (X, τ) is intuitionistic fuzzy generalized β connected between A_1 and B_1.

Proof : Suppose that (X, τ) is not intuitionistic fuzzy generalized β connected between A_1 and B_1, then by Definition 3.12, there exists an intuitionistic fuzzy generalized β open set E in (X, τ) such that $A_1 \subseteq E$ and $E\ q^c\ B_1$. This implies $E \subseteq B_1{}^c$ and $A_1 \subseteq E$ implies $A \subseteq A_1 \subseteq E$. That is $A \subseteq E$. Now let us prove that $E \subseteq B^c$, that is let us prove $E\ q^c\ B$. Suppose that $E\ q\ B$, then by Definition 2.7, there exists an element x in X such that $\mu_E(x) > \gamma_B(x)$ and $\gamma_E(x) < \mu_B(x)$. Therefore $\mu_E(x) > \gamma_B(x) > \gamma^{B1}(x)$ and $\gamma_E(x) < \mu_B(x) < \mu^{B1}(x)$, since $B \subseteq B_1$. Hence $\mu_E(x) > \gamma^{B1}(x)$ and $\gamma_E(x) < \mu^{B1}(x)$. Thus $E\ q\ B_1$. But $E \subseteq B_1{}^c$. That is $Eq^c\ B_1$, which is a contradiction. Therefore $E\ q^c B$. That is $E \subseteq B^c$. Hence (X, τ) is not intuitionistic fuzzy generalized β connected between A and B, which is a contradiction to our hypothesis. Thus (X, τ) must be intuitionistic fuzzy generalized β connected between A_1 and B_1.

Theorem 3.18 : Let (X, τ) be an IFTS and A and B be IFSs in (X, τ). If $A\ q\ B$, then (X, τ) is intitionistic fuzzy generalized β connected between A and B.

Proof : Suppose (X, τ) is not intuitionistic fuzzy generalized β connected between A and B.

Then there exists an intuitionistic fuzzy generalized β open set E in (X, τ) such that $A \subseteq E$ and $E \subseteq B^c$. This implies that $A \subseteq B_c$. That is A q B. But this is a contradiction to our hypothesis. Therefore (X, τ) is must be intuitionistic fuzzy generalized β connected between A and B.

Remark 3.19 : The converse of the above theorem may not be true in general.

Example 3.20 : In Example 3.6, (X, τ) is intuitionistic fuzzy generalized β connected between the IFSs A and B but A is not q-coincident with B, since $\mu_A(x) < \gamma_B(x)$ and $\mu_B(x) < \gamma_A(x)$.

Definition 3.21 : An intuitionistic fuzzy generalized β open set A is an intuitionistic fuzzy regular generalized β open set if A = gβint(gβcal(A)). The complement of an intuitionistic fuzzy regular generalized β open set is called an intuitionistic fuzzy regular generalized β closed set.

Definition 3.22 : An IFTS (X, τ) is called an intuitionistic fuzzy generalized β super connected if there exists no proper intuitionistic fuzzy regular generalized β open set in (X, τ).

Theorem 3.23 : Let (X, τ) be an IFTS, then the following are equivalent :

(i) (X, τ) is an intuitionistic fuzzy generalized β super connected space.

(ii) For every non-zero intuitionistic fuzzy regular generalized β open set A, gβcl(A) = 1~.

(iii) For every intuitionistic fuzzy regular generalized β closed set A wih A ≠ 1~, gβint(A) = 0~.

(iv) There exists no intuitionistic fuzzy regular generalized β open sets A and B in (X, τ) such that A ≠ 0~ ≠ B, $A \subseteq B^c$.

(v) There exists no intuitionistic fuzzy regular generalized β open sets A ≠ 0~ ≠ B, B = $(g\beta cl(A))^c$, A = $(g\beta cl(B))^c$.

(vi) There exists no intuitionistic fuzzy regular generalized β closed sets A and B in (X, τ) such that A ≠ 1~ ≠ B, B = $(g\beta int(A))^c$, A = $(g\beta int(B))^c$.

Proof : (i) ⇒ (ii) Assume that there exists an intuitionistic fuzzy regular generalized β open set A in (X, τ) such that A ≠ 0~ and gβcl(A) ≠ 1~ Now let B = $g\beta int(g\beta cal(A))^c$. Then B is a proper intuitionistic fuzzy regular generalized β open set in (X, τ). But this is a contradiction to the fact that (X, τ) is an intuitionistic fuzzy generalized β super connected space. Therefore gβcl(A) = 1~.

(ii) $\Rightarrow$ (iii) Let $A \neq 1\sim$ be an intuitionistic fuzzy regular generalized β closed set in (X, τ). If $B = A^c$, then B is an intuitionistic fuzzy regular generalized β open set in (X, τ) with $B \neq 0\sim$. Hence $g\beta cl(B) = 1\sim$. This implies $(g\beta cl(B))^c = 0\sim$. That is $g\beta int(B^c) = 0\sim$. Hence $g\beta int(A) = 0\sim$.

(iii) $\Rightarrow$ (iv) Let A and B be two intuitionistic fuzzy regular generalized β open sets in (X, τ) such that $A \neq 0\sim \neq B$ and $A \subseteq B^c$. Since B^c is an intutionistic fuzzy regular generalized β closed set in (X, τ) and $B \neq 0\sim$ implies $B^c \neq 1\sim$. Therefore $B^c = g\beta cl(g\beta int(B^c))$. We have $g\beta int(B^c) = 0\sim$ But $A \subseteq B_c$. Therefore $0\sim \neq A = g\beta int(g\beta cl(A)) \subseteq g\beta int(g\beta cl(B^c)) = g\beta int(g\beta cl(g\beta cl(g\beta int(B^c))) = g\beta int(g\beta cl(g\beta int(B^c))) = g\beta int(B^c) = 0\sim$. A contradiction arises. Therefore (iv) is true.

(iv) $\Rightarrow$ (i) Let $0\sim \neq A \neq 1\sim$ be an intuitionistic fuzzy regular generalized β open set in (X, τ). If we take $B = (g\beta cl(A))$, then B is an intuitionistic fuzzy regular generalized β open set, since $g\beta int(g\beta cl(B)) = g\beta int(g\beta cl(g\beta cl(A))^c) = g\beta int(g\beta int(g\beta cl(A)))^c = g\beta int(A^c) = (g\beta cl(A))^c = B$/ Also we get $B \neq 0\sim$, since otherwise, we have $B = 0\sim$ and this implies $(g\beta cl(A))^c = 0\sim$. That is $g\beta cl(A) = 1\sim$. Hence $A = g\beta int(g\beta cl(A)) = g\beta int(1\sim) = 1\sim$. That is $A = 1\sim$, which is a contradiction. Therefore $B \neq 0\sim$ and $A \subseteq B^c$. But this is a contradiction to (iv). Therefore (X, τ) must be intuitionistic fuzzy generalized β super connected space.

(i) $\Rightarrow$ (v) Let A and B be two intuitionistic fuzzy regular generalized β open sets in (X, τ) such that $A \neq 0\sim \neq B$ and $B = (g\beta cl(A))^c$, $A = (g\beta cl(B))^c$. Now we have $g\beta int(g\beta cl(A)) = g\beta int(B^c) = (g\beta cl(B))^c = A$, $A \neq 0\sim$ and $A \neq 1\sim$, since $A = 1\sim$ then $1\sim = (g\beta cl(B))^c \Rightarrow g\beta cl(B) = 0\sim \Rightarrow B = 0\sim$. But $B \neq 0\sim$. Therefore $A \neq 1\sim \Rightarrow$ A is a proper intuitionstic fuzzy regular generalized β open set in (X, τ), which is a contradiction to (i). Hence (v) is true.

(v) $\Rightarrow$ (i) Let A be an intitionistic fuzzy regular generalized β open set in (X, τ) such that $A = g\beta int(g\beta cl(A))$, $0\sim \neq A \neq 1\sim$. Now take $B = (g\beta cl(A))^c$. In this case we get, $B \neq 0\sim$ and B is an intuitionistic fuzzy regular generalized β open set in (X, τ) and $B = (g\beta cl(A))^c$ and $(g\beta cl(B))^c = (g\beta cl(g\beta cl(A))^c)^c = g\beta int(g\beta cl(A)^c)^c = g\beta int(g\beta cl(A)) = A$. But this is contradiction to (v). Therefore (X, τ) must be an intuitionistic fuzzy generalized β super connected space.

(v) $\Rightarrow$ (vi) Let A and B be two intuitionistic fuzzy regular generalized β closed sets in (X, τ) such that $A \neq 1_\sim \neq B$, $B = (g\beta int(A))^c$, $A = (g\beta int(B))^c$. Taking $C = A^c$ and $D = B^c$, C and D become intuitionistic fuzzy regular generalized β open sets in (X, τ) with $C \neq 0_\sim \neq D$ and $D = (g\beta cl(C))^c$, $C = (g\beta cl(D))^c$, which is a contradiction to (v). Hence (vi) is rue. (vi) $\Rightarrow$ (v) can be proved easily by the similar way as in (v) $\Rightarrow$ (vi).

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IF PARAMETERIZED INTUITIONISTIC FUZZY SOFT SET THEORIES ON DECISIONS—MAKING

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ABSTRACT : In 2010 N. Cagman et. al inroduced the notion of fuzzy parameterized fuzzy soft sets (fpfs-sets) and their operations. In this paper we introduce IF parameterized intuitionistic fuzzy soft sets (ifpifs-sets) and their operations as a generalization of Cagman et. al's work. We also show that IF parameterized intuitionistic fuzzy soft sets theories on decision making problems is more fruitful than that of fuzzy parameterized fuzzy soft sets theories on decision making problems. We define ifpifs-aggregation operator to form ifpifs-set decision making method that allows constructing more powerful decision processes.

Key words : Fuzzy soft set, Intuitionistic fuzzy set, Intuitionistic fuzzy soft set, Fuzzy parameterized fuzzy soft set.
AMS Classification : 54A40.

1. INTRODUCTION

In our real life problems there are situations with the uncertain data that may not be successfully modeled by the classical mathematics. There are some mathematical tools for dealing with uncertainties-they are fuzzy set theory introduced by Zadeh (1965)[12] and soft set theory initiated by Molodtsov (1999)[9], that are related to our work. The aim of this paper is to construct ifpifs-set decision making problem and to solve the problem with IF parameterized intuitionistic fuzzy soft set theories.

We now give some ready references for further discussion :

Definition 1.1[12] Let U be a universe. A fuzzy set X over U is defined by a function μ_X representing a mapping $\mu_X:U\to[0,1]$. Here μx is called the membership function of X and the value $\mu_X(u)$ is called the grade of membership of $u\in U$. The value represents the degree of u belonging to the fuzzy set X. Thus a fuzzy set X over U can be represented as follows:

$$X = \{(\mu_X(u)/u) : u \in U, \ \mu_X(u) \in [0,1]\}$$

After the introduction of concept of fuzzy sets by Zadeh [12] several researches were conducted on the generalization of the notion of the fuzzy set. The idea of "intuitionistc fuzzy set" was first published by Atanassov [1].

Definition 1.2[1] Let a set E be fixed. An intuitionistic fuzzy set or IFS 'A' in E is an object having the form $A = \{ (x, \mu_A(x), \nu_A(x) : x \in E) \}$ where the function $\mu_A : E \to I = [0, 1]$ & $v_A : E \to I = [0, 1]$ define the degree of membership and non-membership respectively of the element $x \in E$ to the set A & for every $x \in E$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. The rest part $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the indeterministic part of x and $0 \leq \pi_A(x) \leq 1$.

Definition 1.3[1] Let X be a nonempty set and the IF sets A and B be in the form $A = \{ (x, \mu_A(x), \nu_A(x) : x \in X) \}$, $B = \{ (x, \mu_B(x), \nu_B(x) : x \in X) \}$.

Then

(a) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$.

(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

(c) $A^c = \{ (x, \nu_A(x), \mu_A(x) : x \in X) \}$.

(d) $A \cap B = \{(x, \mu_A(x) \cap \mu_B(x), \nu_A(x) \cup \nu_B(x) : x \in X)\}$.

(e) $A \cup B = \{(x, \mu_A(x) \cup \mu_B(x), \nu_A(x) \cap \nu_B(x) : x \in X)\}$.

In 1999, Molodtsov [9], has inroduced the concept of soft set in the following way :

Definition 1.4 [9] Let U be an initial universal set, and E be the set of parameters. Let P(U) denotes the power set of U & $A \subseteq E$, then the pair $\xi = (F, A)$ is called a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, F(e) may be considered as a sets of e-approximate elements of the soft set (F, A), where $F : A \to P(U)$.

Definition 1.5[5] Let U be an initial universe, E be the set of all parameters, $A \subseteq E$ and $\eta_A(x)$ be a fuzzy set over U for all $x \in E$. Then a fuzzy soft set (fs-set) Γ_A over U is a set defined by a function η_A representing a mapping $\eta_A : E \to P(U)$ such that $\eta_A(x) = \phi$ if $x \notin A$. Here

η_A is called the fuzzy approximate function of the fs-set Γ_A over U and Γ_A can be represented by the set of ordered pairs

$$\Gamma_A = \{(x, \eta_A(x)) : x \in E, \eta_A(x) \in P(U)\}$$

Definition 1.6[3] Let U be an initial universe, P(U) be the power set of U, E be the set of all parameters and X be a fuzzy set over E with the membership function $\mu_X : E \to [0,1]$. Then the fps-set F_X over U is a set defined by the function f_X representing a mapping

$$f_X : E \to P(U) \text{ such that } f_X(x) = \phi \text{ if } \mu_X(x) = 0$$

Here f_X is called approximate function of the fps-set F_X, and the value f_X is a set called x-element of the fps-set for all $x \in E$. Thus an fps-set F_X over U can be represented by the set of pairs.

$$F_X = \{(\mu_X(x) / x, f_X(x)) : x \in E, f_X(x) \in P(U), \mu_X(x) \in [0, 1]\}.$$

For example let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be a universal set and $E = \{x_1, x_2, x_3, x_4\}$ be a set of parameters.

If $X = \{0.2/x_2, 0.5/x_3, 1/x_4\}$ and $f_X(x_2) = \{u_2, u_4\}$, $f_X(x_3) = \phi$, $f_X(x_4) = U$;

The fps-set F_X is written as $F_X = \{(0.2 / x_2, \{u_2, u_4\}), (1/x_4, U)\}$.

Now the approximate functions of fuzzy parameterized fuzzy soft set (fpfs-set) are defined from fuzzy parameters set to the fuzzy subsets of universal set.

Here we use $\Gamma_X, \Gamma_Y, \Gamma_Z$,.....etc for fs-sets and η_X, η_Y, η_Z....etc for their fuzzy approximate functions respectively.

Definition 1.7[3] Let U be initial universe, E be the set of parameters and X be a fuzzy set over E with the membership function $\mu_X : E \to [0, 1]$ and $\eta_X(x)$ be a fuzzy set over U for all $x \in E$. Then a fpfs-set Γ_X over U is a set defined by a function $\eta_x(x)$ represents a mapping $\eta_X : E \to P(U)$ such that $\eta_X(x) = \phi$ if $\mu_X(x)=0$. Here η_X is called the fuzzy approximae function of the fpfs-set Γ_X and the value $\eta_X(x)$ is a fuzzy set called x-element of the fpfs-set for all $x \in E$. Thus a fpfs set Γ_X over U can be represented by the set of ordered pairs

$\Gamma_X = \{(\mu_X(x)/x, \eta_X(x)) : x \in E, \eta_X(x) \in P(U), \mu_X(x) \in [0,.1]\}$.

For example assume that $U = \{u_1, u_2, u_3, u_4, u_5\}$ is a universal set and $E=\{x_1, x_2, x_3, x_4\}$ is a set of parameters. If $X = \{0.2/x_2, 0.5/x_3, 1/x_4\}$ and $\eta_X(x_2) = \{0.5/u_1, 0.3/u_3\}$, $\eta_X(x_3) = \phi$, $\eta_X(x_4) = U$, then the fpfs-set Γ_X is written as

$$\Gamma_X = \{(0.2/x_1, \{0.5/u_1, 0.3/u_3\}), (1/x_4, U)\}.$$

2. IF Parametrized Intuitionistic Fuzzy Soft Set theories on Decisionmaking.

In this section we introduce the concept of IF Parametrized Intuitionistic Fuzzy Soft Set (ifpifs-set) and their operations with examples. The approximae functions of ifpifs-set are defined from intuitionistic fuzzy parameter (IFP) set to the intuitionistic fuzzy subsets of universal set.

Definition 2.1 Let U be an initial universe, E be the set of IF parameters and X be a IF set over E with the membership function $\mu_X : E \to [0,1]$ and non-membership function $\gamma_X : E \to [0,1]$ where $0 \leq \mu_X(x) + \gamma_X(x) \leq 1$ and $\eta_X(x) = \{(A_X(x), B_X(x))/u\}$ be an IF set over U for all $x \in E$, $A_X(x), B_X(x) \in [0,1]$ and $u \in U$. Then an ifpifs-set Γ_X over U is a set defined by a function $\eta_X(x)$ represents a mapping $\eta_X : E \to P(U)$ such that $\eta_X(x) = \phi$ if $\mu_X(x) = 0$ and $\gamma_X(x) = 1$. Here ηx is called the IF approximate function of the ifpifs-set Γ_X and the value $\eta_X(x)$ is an IF set called x-element of the ifpifs-set for all $x \in E$. Thus an ifpifs set Γ_X over U can be represented by the set of ordered pairs

$$\Gamma_X = \{(\mu_X(x), \gamma_X(x))/x), \eta x(x)) : x \in E, \eta_X(x) \in P(U), \mu_X(x), \gamma_X(x) \in [0,1]\}.$$

$= \{((\mu_X(x), \gamma_X(x))/x, ((Ax(x), Bx(x)/u)\} : x \in E, \eta x(x) = (A_X(x), B_X(x))/u) \in P(U), \mu_X(x), \gamma_X(x), Ax(x), Bx(x) \in [0,1]\}$. P(U) is the family of intuitioistic fuzzy subsets of U. We denote the sets of all ifpifs-sets over U by IFPIFS(U).

Example 2.2 Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ is a universal set and $E=\{x_1, x_2, x_3, x_4\}$ is a set of IF parameters. If $X = \{(0.2, 0.7)/x_2, (0.5, 0.3)/x_3, (1, 0)/x_4\}$ and $\eta_X(x_2) = \{0.5, 0.4)/u_1, (0.3, 0.6)/u_3\}$, $\eta_X(x_3) = \phi$, $\eta_X(x_4) = U$, then the fpfs-set Γ_X is written as

$$\Gamma_X = \{(0.2, 0.7)/x_1, \{(0.5, 0.4)/u_1, (0.3, 0.6)/u_3\}), ((1,0)/x_4, U)\}.$$

Definition 2.3 Let $\Gamma_X \in$ IFPIFS(U). If, $\eta_X(x) = U$ for all $x \in E$, i.e., $\mu_X(x) = 1$, $\gamma_X(x) = 0$ then Γ_X is called X-universal ifpifs-set, denoted by Γ_X.

If X = E then the X-universal ifpifs-set (Γ_X) is called universal ifpifs-set denoted by Γ_E.

Definition 2.4 Let $\Gamma_X \in$ IFPIFS(U). If, $\eta_X(x) = \Phi$ for all $x \in E$, i.e., $\mu_X(x) = 0$, $\gamma_X(x) = 1$ then Γ_X is called X-empty ifpifs-set, denoted by $\Gamma_{\Phi X}$.

If $X = \Phi$ then the X-empty ifpifs-set ($\Gamma_{\Phi X}$) is called empty ifpifs-set denoted by Γ_Φ.

Example 2.5 Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ is a universal set and $E=\{x_1, x_2, x_3, x_4\}$ is a set of IF parameters. If $X = \{(0.2, 0.7)/x_2, (0.5, 0.3)/x_3, (1, 0)/x_4\}$ and $\eta x(x_2) = \{0.5, 0.4)/u_1, (0.3, 0.6)/u_3\}$, $\eta x(x_3) = \phi$, $\eta x(x_4) = U$, then the ifpifs-set Γx is written as $\Gamma x = \{(0.2, 0.7)/x_2, \{(0.5, 0.4)/u_1, (0.3, 0.6)/u_3\}), ((1,0)/x_4, U)\}$. Now, if $Y = \{(1, 0)/x_1, (0.7, 0.2)/x_4\}$ and $\eta x(x_1) = \Phi$, $\eta x(x_4) = \Phi$, then ifpifs-set Γ_Y is a Y-empty ifpifs-set, i.e., $\Gamma_Y = \Gamma_{\Phi Y}$.

If $Z = \{(1, 0)/x_1, (1,0)/x_2)$, $\eta x(x_1) = U$, $\eta x(x_2) = U$, then the ifpifs-set Γ_Z is Z-universal ifpifs-set.

If $X = \Phi$, then ifpifs-set Γx is an empty set i.e., $\Gamma_X = \Gamma_\Phi$.

If X = E and $\eta x\ (x_1) = U$ for all $x_i \in E$ (i = 1,2,3,4) then ifpifs-set Γx is a universal ifpifs-set, i.e. Γ_X

$= \Gamma_E$.

Definition 2.6 Let $\Gamma_X, \Gamma_Y \in$ IFPIFS(U). Then Γ_X is an ifpifs-subset of Γ_Y, denoted by $\Gamma x \sqsubseteq \Gamma_Y$ if $\mu x(x) \leq \mu_Y(x)$, $(\gamma x(x) \geq \gamma_Y(x)$ and $\eta x(x) = ((Ax(x), Bx(x))/u) \subseteq \eta y(x) = ((A_Y(x), B_Y(x)/u$ for all $x \in E$.

Definition 2.7 Let $\Gamma_X, \Gamma_Y \in$ IFPIFS(U). Then Γ_X and Γ_Y are ifpifs-equal of Γ_Y, written as $\Gamma x = \Gamma_Y$ if $\mu x(x) = \mu_Y(x)$, $(\gamma x(x) = \gamma_Y(x)$ and $\eta x(x) = ((Ax(x), Bx(x))/u) = \eta y(x) = ((A_Y(x), B_Y(x)/u$ for all $x \in E$.

Definition 2.8 Let $\Gamma_X, \Gamma_Y, \Gamma_Z \in$ IFPIFS(U). Then

(i) $(\Gamma x = \Gamma_Y$ and $\Gamma_Y = \Gamma_Z) \Rightarrow \Gamma_X = \Gamma_Z$.

(ii) ($\Gamma x \subseteq \Gamma_Y$ and $\Gamma_Y \subseteq \Gamma_X$) $\Rightarrow \Gamma x = \Gamma_Y$.

Proof : The proofs are trivial.

Definition 2.9 Let $\Gamma_X \in$ IFPIFS(U), then the complement of Γ_X denoted by $\Gamma_X{}^c$ is defined by $\Gamma_X{}^c = \{\gamma x(x), \mu x(x))/x, \eta x^c(x)) : x \in E, \eta x^c(x) \in P(U), \mu x(x), \gamma x(x) \in [0,1]\}$, where $\eta x^c(x) = (\eta x(x))^c = U - \eta x(x) = \{U-(A_X(x), Bx(x))/u)\} = U \cap (Bx(x), Ax(x))/u) = (Bx(x), Ax(x))/u)$.

Definition 2.10 (a) Let $\Gamma_X, \Gamma_Y \in$ IFPIFS(U). Then union of Γ_X and Γ_Y, denoted by $\Gamma_X \cup \Gamma_Y$ is defined by $\Gamma_X \cup \Gamma_Y = \{(\mu x \cup Y(x), \gamma x \cap Y(x)/x), \eta x \cup Y(x)\}$

$= [(\max\{\mu x(x), \mu_Y(x)\}, \min\{\gamma x(x), \gamma y(x)\})/x, (\eta x(x) \cup \eta y(x))]$

$= [(\max\{\mu x(x), \mu_Y(x)\}, \min\{\gamma x(x), \gamma y(x)\})/x, \{(A_X(x), Bx(x))/u)\} \cup (A_Y(x)), B_Y(x))/u\}]$.

$= [\max\{\mu x(x), \mu_Y(x)\}, \min\{\gamma x(x), \gamma y(x)\})/x, \{\max (A_X(x), (A_Y(x)), \min Bx(x)), B_Y(x))/u\}]$.

for all $x \in E$.

(b) The intersection of Γ_X and Γ_Y denoted by $\Gamma_X \cap \Gamma_Y$ is defined by $\Gamma_X \cap \Gamma_Y = \{(\mu x \cap Y(x), \gamma x \cup Y(x)/x), \eta x \cap Y(x)\}$

$= [(\min\{\mu x(x), \mu_Y(x)\}, \max\{\gamma x(x), \gamma y(x)\})/x, (\eta x(x) \cap \eta y(x))]$

$= [(\min\{\mu x(x), \mu_Y(x)\}, \max\{\gamma x(x), \gamma y(x)\})/x, \{(A_X(x), Bx(x))/u)\} \cap (A_Y(x)), B_Y(x))/u\}]$.

$= [\min\{\mu x(x), \mu_Y(x)\}, \max\{\gamma x(x), \gamma y(x)\})/x, \{\min (A_X(x), (A_Y(x)), \max Bx(x)), B_Y(x))/u\}]$.

for all $x \in E$.

Result 2.11 Let $\Gamma_X, \Gamma_Y, \Gamma_Z \in$ IFPIFS(U). Then

(i) $\Gamma_X \cup \Gamma_X = \Gamma_X$ and $\Gamma_X \cap \Gamma_X = \Gamma_X$.

(ii) $\Gamma_{\Phi X} \cup \Gamma_X = \Gamma_X$ and $\Gamma_{\Phi X} \cap \Gamma_X = \Gamma_X$.

(iii) $\Gamma_\Phi \cup \Gamma_X = \Gamma_X$ and $\Gamma_\Phi \cap \Gamma_X = \Gamma_\Phi$.

(iv) $\Gamma_X \cup \Gamma_E = \Gamma_E$ and $\Gamma_X \cap \Gamma_E = \Gamma_X$.

(v) $\Gamma_X \cup \Gamma_Y = \Gamma_Y \cup \Gamma_X$ and $\Gamma_X \cap \Gamma_Y = \Gamma_Y \cap \Gamma_X$.

(vi) $(\Gamma_X \cup \Gamma_Y) \cup \Gamma_Z = \Gamma_X \cup (\Gamma_Y \cup \Gamma_Z)$ and $(\Gamma_X \cap \Gamma_Y) \cap \Gamma_Z = \Gamma_X \cap (\Gamma_Y \cap \Gamma_Z)$.

It is to be noted that if $\Gamma_X \neq \Gamma_E$ or $\Gamma_X \neq \Gamma\Phi$ then $\Gamma_X \cup \Gamma_X^c \neq \Gamma_E$ and $\Gamma_X \cap \Gamma_X^C \neq \Gamma\Phi$

Theorem 2.12 Let $\Gamma_X, \Gamma_Y \in$ IFPIFS(U). Then De Morgan's laws are valid

(i) $(\Gamma_X \cup \Gamma_Y)^C = \Gamma_X^C \cap \Gamma_Y^C$.

(ii) $(\Gamma_X \cap \Gamma_Y)^C = \Gamma_X^C \cup \Gamma_Y^C$.

Proof : (i) For all $x \in E$,

$(\mu(X \cup Y)^c(x), \gamma(X \cap Y)^c(x)) = \{1-\mu_{X \cup Y}(x), 1-\gamma_{X \cap Y}(x)\} = (1-\max[\mu_X(x), \mu_Y(x)]\}, (1-\min[\gamma_X(x, \gamma_Y(x)])\}$

$= \min\{1-\mu_X(x), 1-\mu_Y(x)\}, \max\{1-\gamma_X(x), 1-\gamma_Y(x)\} = \min\{\mu_{X^c}(x), \mu_{Y^c}(x)\}, \max\{\gamma_{X^c}(x), \gamma_{Y^c}(x)\}$

$= \mu_{X^c \cap Y^c}(x), \gamma_{X^c \cup Y^c}(x)$ and

$\eta(X \cup Y)^c(x) = \eta(X \cup Y)^c(x) = (\eta_X(x) \cup \eta_Y(x)^c = (\eta_X(x))^c \cap (\eta_Y(x)^c = \eta_{X^c}(x) \cap \eta_{Y^c}(x)$

$= (B_X(x), A_X(x)/u) \cap (B_Y(x), A_Y(x)/u) = \{\min(B_X(x), B_Y(x), \max (A_X(x), A_Y(x))\}/u$

$= (B_{X \cap Y}(x), A_{X \cup Y}(x))/u = \eta_{X^c \cap Y^c}(x)$. Hence the result.

Likewise the proof of (ii) can be made easily.

Theorem 2.13 Let $\Gamma_X, \Gamma_Y, \Gamma_Z \in$ IFPIFS(U). Then

(i) $\Gamma_X \cup (\Gamma_Y \cap \Gamma_Z) = (\Gamma_X \cup \Gamma_Y) \cap (\Gamma_X \cup \Gamma_Z)$

(ii) $\Gamma_X \cap (\Gamma_Y \cup \Gamma_Z) = (\Gamma_X \cap \Gamma_Y) \cup (\Gamma_X \cap \Gamma_Z)$

Proof : For all $x \in E$,

$(\mu(X \cup (Y \cap Z))(x), \gamma(X \cap (Y \cup Z))(x) = \{(\max[\mu_X(x), \mu_{Y \cap Z}(x)]), (\min[\gamma_X(x, \gamma_{Y \cup Z}(x)])\}$

$= \max \{\mu_X(x), \min(\mu_Y(x), \mu_Z(x))\}, \min \{\gamma_X(x), \max(\gamma_Y(x), \eta_Z(x))\}$

$= \min \{\max(\mu_X(x), \mu_Y(x)), \max(\mu_X(x))\}, \max \{\min(\gamma_X(x), \gamma_Y(x)), \min(\gamma_X(x), \gamma_Z(x))\}$

$= \mu_{(X \cup Y) \cap (X \cup Z)}(x), \gamma_{(X \cap Y) \cup (X \cap Z)}(x)$ and

$\eta_{(X \cup (Y \cap Z))}(x) = \eta_X(X) \cup \eta_{(Y \cap Z)}(x) = \eta_X(X) \cup \{\eta_Y(x) \cap \eta_Z(x)\} = \{\eta_X(x) \cup \eta_Y(x)) \cap (\eta_X(x)\} \cup \eta_Z(x)) = (\eta_{X \cup Y}(x) \cap \eta_{X \cup Z}(x)) = (\eta_{(X \cup Y) \cap (X \cup Z)})(x)$.

3. ifpifs-aggregation operator

In this section we define an aggregate IF set of a ifpifs-set. We also define ifpifs-aggregation operator that produce an aggregate IF set from an ifpifs-set and its IF parameter set.

The concept of IF Parameterized Intuitionistic Fuzzy Soft Set (ifpifs-set) and their operations are given in section 2.

We define an aggregate IF set of a ifpifs-set. We also define ifpifs-aggregation operator that produce an aggregate IF set from an ifpifs-set and its IF parameter set.

Definition 3.1 Let $\Gamma_X \in$ IFPIFS(U), then ifpifs-aggregation operator, denoted by IFPIFSagg, is defined by IFPIFSagg : P(E) × IFPIFS(U) → P(U), where IFPIFSagg (X, Γ_X) = Γ_X^* and $\Gamma_X^* = \{ \mu \Gamma_X^*(u), \gamma \Gamma_X^*(u))/u : u \in U\}$, which is an IF set over U. The value Γ_X^* is called aggregate IF set of the set Γ_X. Here the membership degree $\Gamma_X^*(u)$ of u and the non-membership degree of $\gamma \Gamma_X^*$ (u) of u are defined as $(\mu \Gamma_X^*(u), \gamma \Gamma_X^*(u))$

$$= 1/|E| \sum_{x \in E} \{\mu_X(x) A_X(x), \gamma_X(x) B_X(x)\}$$

where $\eta_X(x) = (A_X(x), B_X(x))/u$, and $|E|$ is the cardinality of E.

We now construct an ifpifs-set decision making method by the following steps—

(i) First construct an ifpifs-set Γ_X over U.

(ii) Find the aggregate IF-set Γ_X^* of Γ_X.

(iii) Find the maximum membership grade of $\mu \Gamma_X^*(u)$, and observe the values of $\gamma \Gamma_X^*(u)$.

Example 3.2 Now we give an example for the above concept. Assume that an office wants to file a post, there are eight candidates. So, $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$. The recruiting

committee consider a set of parameters $E = \{x_1, x_2, x_3, x_4, x_5\}$. The parameters x_i, ($i = 1,2,3,4,5$) stands for 'experience', 'computer knowledge', 'young age', 'good speaking' and 'friendly', respectively. After interview each candidate is evaluated from point of view of goals and the constraint according to a chosen subset $X = \{(0.5,0.4)/x_2, (0.9,0.1)/x_3, (0.6,0.3)/x_4\}$ of E and finally the committee constructs the following ifpifs-set over U

$\Gamma x = [(0.5,0.4)/x_2, \{(0.3,0.6)/u_2, (0.4,0.5)/u_3, (0.1,0.9)/u_4, (0.9,0.1)/u_5, (0.7,0.2)/u_7\}), ((0.9,0.1)/x_3, \{(0.4,0.5)/u_1, (0.4,0.6)/u_2, (0.9,0.1)/u_3, (0.3,0.6)/u_4\}), ((0.6,0.3)/x_4, \{(0.2,0.7/u_1, (0.5,0.4)/u_2, (0.1,0.8)/u_5, (0.7,0.3)/u_7, (1.0)/u_8\}]$.

Thus the aggregate IF set

$\Gamma x^* = \{(0.096,0.052)/u_1, (0.162,0.084)/u_2, (0.202,0.042)/u_3, (0.064,0.084)/u_4, (0.102,0.056)/u_5, (0.154,0.034)/u_7, (0.12,0)/u_8\}$.

Finally the largest membership grade $\mu\, \Gamma x^*(u)=0.202$ has been chosen, and it is to be observed that the corresponding value of $\gamma\, \Gamma x^*(u) = 0.042$, which means that the candidate u_3 is selected for the post. Now the question arises 'what is the role of $\gamma\, \Gamma x^*(u)$'? Here is the answer- In case the largest membership grade $\mu\, \Gamma x^*(u)$ are same for some candidates then chose the candidate having the smallest non-membership grade $\gamma\, \Gamma x^*(u)$ among them.

Example 3.3 Considering the example 2.3, we construct the ifpifs-set over U as

$\Gamma x = [(0.5,0.4)/x_2, \{(0.4,0.6)/u_2, (0.4,0.5)/u_3, (0.1,0.9)/u_4, (0.9,0.1)/u_5, (0.7,0.2)/u_7\}), ((0.9,0.1)/x_3, \{(0.4,0.5)/u_1, (0.4,0.6)/u_2, (0.9,0.1)/u_3, (0.3,0.6)/u_4\}), ((0.6,0.3)/x_4, \{(0.2,0.7/u_1, (0.6,0.4)/u_2, (0.1,0.8)/u_5, (0.7,0.3)/u_7, (1.0)/u_8\}]$.

Thus the aggregate IF set

$\Gamma_X^* = \{(0.096,0.052)/u_1, (0.202,0.084)/u_2, (0.202,0.042)/u_3, (0.064,0.084)/u_4, (0.102,0.056)/u_5, (0.154,0.034)/u_7, (0.12,0)/u_8\}$.

Here u_2 and u_3 have the same largest membership grade $\mu\, \Gamma_X^*(u)=0.202$, but we observe that the minimum non-membership value $\gamma\, \Gamma_X^*(u) = 0.042$, between u_2 and u_3. So u_3 is selected for the post.

Example 3.4 Cagman et.al[3], gave the fpfs-set decision making problem by taking $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$. With set of fuzzy parameters $E = \{x_1, x_2, x_3, x_4, x_5\}$ and construct fpfs-set $\Gamma x = [(0.5)/x_2, \{(0.3)/u_2, (0.4)/u_3, (0.1)/u_4, (0.9)/u_5, (0.7)/u_7\}), ((0.9)/x_3, \{(0.4)/u_1, (0.4)/u_2, (0.9)/u_3, (0.3)/u_4\}), ((0.6)/x_4, \{(0.2)/u_1, (0.5)/u_2, (0.1)/u_5, (0.7)/u_7, (1)/u_8\})]$, where $X = \{(0.5)/x_2, (0.9)/x_3, (0.6)/x_4\}$ of E.

Thus the aggregate IF set

$\Gamma x^* = \{(0.096)/u_1, (0.162)/u_2, (0.202)/u_3, (0.064)/u_4, (0.102)/u_5, (0.154/u_7, (0.12)/u_8\}$.

Finally the candidate u_3 having the largest membership grade had been chosen for the job.

But we observe that if $X = \{(0.5)/x_2, (0.9)/x_3, (0.6)/x_4\}$ of E and if $\Gamma x = [(0.5)/x_2, \{(0.3)/u_2, (0.4)/u_3, (0.1)/u_4, (0.9)/u_5, (0.7)/u_7\}), ((0.9)/x_3, \{(0.4)/u_1, (0.5)/u_2, (0.9)/u_3, (0.3)/u_4\}), ((0.6)/x_4, \{(0.2)/u_1, (0.6)/u_2, (0.1)/u_5, (0.7)/u_7, (1)/u_8\})]$ then the aggregate IF set

$\Gamma x^* = \{(0.096)/u_1, (0.202)/u_2, (0.202)/u_3, (0.064)/u_4, (0.102)/u_5, (0.154/u_7, (0.12)/u_8\}$. Then u_2 and u_3 have the same largest membership value. So, it is not possible to select one between the two candidates. In this case we need the ifpifs-set theories.

Conclusion : Cagman et.al[3], gave the fpfs-set decision making problem. Here we construct ifpifs-set decision making problem, which is more fruitful incase when the candidates having the same largest membership grade $\mu\ \Gamma x^*(u)$.

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SEQUENTIAL INTERIOR OPERATORS

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ABSTRACT : In this communication, operators from $(P(X))^N$ into itself that behave like usual interior operator and their components are introduced. It is also observed what do the operators and their components produce and what are the relations among themselves. Two dimensional contraction (in some sense) of members of $(P(X))^N$ plays the main role in this article.

Key words and phrases : Sequential sets, sequential interior operators, components of sequential interior operators.

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1. INTRODUCTION

As in case of Kuratowski closure operaor, $k\Omega$-closure operaor [8] generates sequential topology [4,7] on the underlying set. On the other hand, components of monotonic sequential operators [9] yield generalized open sets [1, 2, 3, 5, 6]. However, in this paper we introduce the concept of sequential interior operators and study some properties of them analogous to that of $K\Omega$-closure operators. For this we provide some basic definitions and results as ready references that will be used in sequel. As in [4, 7, 8, 9] any sequence $A(s) = \{A_n\}$, where $A_n \subset X$ for all $n \in$ N is called a sequential set in X and A_n is called n^{th} component of A(s); we write $P_n(A(s)) = A_n$. Thus sequential sets in X are precisely the members of $(P(X))^N$. If each $A_n = X$, $n \in N$, then the corresponding sequential set is denoted by $X(s)$ and $\Phi(s)$ denotes the sequential set having each term equal to Φ. If $F(s) \in (P(X))^N$ and $A \in P(X)$ the ${}_nF_A(s)$ denotes the sequential set obtained from $F(s)$ replacing n^{th} component of it by A. A sequential set $A(s) = \{A_n\}$ is said to be contained in a sequential set $B(s)$ if $A_n \subset B_n$ for all $n \in N$ and it is denoted by $A(s) \subset B(s)$ or $B(s) \supset A(s)$. If $A(s) \subset B(s)$ and $B(s) \subset A(s)$ then $A(s)$ and $B(s)$ are said to be identical and write $A(s) = B(s)$. The union and intersection of the sequential sets $A(s) = \{A_n\}$ and $B(s) = \{B_n\}$ in X are defined as $A(s) \cup B(s) = \{A_n \cup B_n\}$ and $A(s)$

$\cap B(s) = \{A_n \cap B_n\}$ respectively. The complement of $B(s)$ in $A(s)$ is denoted by $A(s)-B(s)$ and is defined by $A(s)-B(s) = \{A_n-B_n\}$. $X(s)-A(s)$ is called the complement of $A(s)$ and it is denoted by $A^c(s)$. Let X be a non-void set. A subset τ of $(P(X))^N$ is called a sequential topology (briefly ST) on X if

i. τ contains $X(s)$ and $\Phi(s)$,

ii. τ is closed under arbirary union,

iii. τ is closed under finite intersection

and ordered pair (X, τ) is called a sequential topological space (STS). The members of τ are called τ–open sequential sets or open sequential sets simply, if no confusion arises. A sequential set A(s) is said to be closed if its complement is open. For any topology D on X, D^N forms an ST on X, called the ST generated by D and is denoted by $\tau < D >$. In an STS (X, τ) the collection $D_n(\tau)$ of the n^{th} components of members of τ forms a topology on X and is called the n^{th} component topology of τ on X. If $A(s) = \{A_n\} \in \tau$, then $A_n \in D_n(\tau)$ for each $n \in$ N, but the converse is not necessarily true. The closure (resp. interior) of a sequential set $A(s)$, denoted by $\overline{A(s)}$ or $Cl_\tau(A(s))$ (resp. $A(s)^o$ or $Int_\tau(A(s))$, is defined to be the intersection (resp. union) of all closed (resp. open) sequential sets containing (resp. contained in) $A(s)$. In an STS (X, τ) the following hold : (1) $A(s) \subset \overline{A(s)}$, (2) $\overline{X(s)} = X(s)$ and $\overline{\Phi(s)} = \Phi(s)$, (3) $\overline{\overline{A(s)}} = \overline{A(s)}$, (4) $\overline{A(s) \bigcup B(s)} = \overline{A(s)} \bigcup \overline{B(s)}$, (5) $\overline{A(s) \bigcap B(s)} \subseteq \overline{A(s)} \bigcap \overline{B(s)}$, (6) A sequential point $p = (x, P) \in \overline{A(s)}$ if and only if every weak neighborhood of p and $A(s)$ are intersecting, (7) For any sequential set $A(s)$, $\overline{A(s)} = A(s) \cup A'(s)$, (8) $A(s)^o \subset A(s)$, (9) $\Phi(s)^o = \Phi(s)$ and $X(s)^o = X(s)$, (10) $(A(s)^o)^o = A(s)^o$, (11) $(A(s) \cap B(s)^o = A(s)^o \cup B(s)^o$, (12) $A(s)^o \cup B(s)^o \subsetneq (A(s) \cup B(s))^o$, (13) $\overline{A(s)} = X(s) - (X(s) - A(s))^o$.

2. Sequential interior operators, their components and relative interior operaors.

Let's begin with the operator M in the following **Example 2.1** that makes sense to define sequential interior operators :

Example 2.1. Consider the operator $M : (P(X))^N \to (P(X))^N$ Defined by $M(A(s)) =$ $= \{\bigcap_{i=n}^{\infty} A_i\}$ $\forall A(s) = \{A_n\} \in (P(X))^N$. It is not hard to check that

(i) $M(A(s)) \subset A(s)$ for all $A(s) \in (P(X))^N$

(ii) $M(X(s)) = X(s)$

(iii) $M(M(A(s))) = M(A(s))$ for all $A(s) \in (P(X))^N$

(iv) $M(\bigcap_{i=1}^{n} A_i(s)) = \bigcap_{i=1}^{n} M(A_i(s))$, where $A_i(s) \in (P(X))^N$, $i = 1(1)n$.

Considering this as a model, we define sequential interior operator as follows :

Definition 2.1. Any operator $I : (P(X))^N \to (P(X))^N$ possessing the afore-mentioned properties (i), (ii), (iii) and (iv) of the operator M in **Example 2.1** is called a sequential interior operator on $(P(X))^N$.

Theorem 2.1. *Let* $I : (P(X))^N \to (P(X))^N$ *be a sequential interior operator. Then* $\tau_I = \{A(s) : A(s) \in (P(X))^N, I(A(s)) = A(s)\}$ = *forms an ST on X and* $I(A(s)) = A(s)^o$, *for all* $A(s) \in (P(X))^N$, *where* $A(s)^o$ *is the interior of* $A(s)$ *in* (X, τ_I).

Proof : Proof is straightforward.

Definition 2.2. $\tau_I = \{A(s) : A(s) \in (P(X))^N, I(A(s)) = A(s)\}$ is called the sequential topology (ST) generated or induced by the sequential interior operator $I : (P(X))^N \to (P(X))^N$.

Definition 2.3. Let X be a nonvoid set and $I : (P(X))^N \to (P(X))^N$ be a sequential interior opeator. A function $I_n : P(X) \to P(X)$ defined by $I_n(A) = P_n(I({}_nX_A(s)))$ $\forall A \subset X$ is called the n^{th} component of I.

For the identity operator I on $(P(X))^N$ which is obviously a sequential interior operator, the n^{th} component of the interior of a sequential set in (X, τ_I) is equal to the interior of the n^{th} component of that sequential set in $(X, D_n(\tau_I))$ but, in case of Example 2.1 they are not equal.

Example 2.2 n^{th} component $I_n : P(X) \to P(X)$ of both the identity operator on $(P(X))^N$ and the seqneutial interior operator in **Example 2.1** is defined by

$$I_n(A) = A, \text{ for all } A \subset X.$$

Theorem 2.2. *For any sequential interior operator* $I : (P(X))^{\mathrm{N}} \to (P(X))^{\mathrm{N}}$, *the operator* $C : (P(X))^{\mathrm{N}} \to (P(X))^{\mathrm{N}}$ *defined by*

$$C(A(s)) = X(s) - I(X(s) - A(s)) \ \forall \ A(s) \in (P(X))^{\mathrm{N}}$$

is a $K\Omega$*-closure operator [8] and*

$$C_n(A) = X - I_n(A) \ \forall \ A \subset X \text{ and } \forall \ n \in N.$$

Proof : We leave proof of this theorem for readers, rather we prove the following.

Theorem 2.3. *Let* X *be a nonvoid set. If* $I : (P(X))^{\mathrm{N}} \to (P(X))^{\mathrm{N}}$ *is a sequential interior operator then each component* $I_n : P(X) \to P(X)$, $n \in N$ *is an interior operator. Also* $D_n(\tau_I) = I_n \tau$, *where* τ_I *is the ST on* X *induced by the seqneutial interior operator* $I : (P(X))^{\mathrm{N}} \to (P(X))^{\mathrm{N}}$ *and* $I_n\tau$ *is the topology on* X *induced by the component* $I_n : P(X) \to P(X)$ *of* I, $n \in N$.

Proof. Let $A \in P(X)$. Then, $I({}_nX_A(s)) \subset {}_nX_A(s) \Rightarrow I_n(A) \subset A$. By definition it follows that X is a fixed point of I_n. Now,

$$
\begin{aligned}
& {}_nX_{In}(A)(s) \subset {}_nX_A(s) \\
\Rightarrow \quad & I({}_nX_{In}(A)(s)) \subset I({}_nX_A(s)) \\
\Rightarrow \quad & I_n(I_n(A)) \subset I_n(A).
\end{aligned}
$$

Also, $I(I({}_nX_A(s))) = I({}_nX_A(s))$.

Clearly, $I({}_nX_A(s)) \subset {}_nX_{In}(A)(s)) \Rightarrow I(I({}_nX_A(s))) \subset I({}_nX_{In}(A)(s))$. Hence,

$I({}_nX_A(s)) \subset I({}_nX_{In}(A)(s)) \Rightarrow I_n(A) \subset I_n(I_n(A))$.

Therefore we have, $I_n(I_n(A)) = I_n(A)$.

Let $A_i \in (P(X))$, $i = 1(1)n$. Now,

$$
\begin{aligned}
& I(\bigcap_{i=1}^{n} {}_nX_{A_i}(s)) = \bigcap_{i=1}^{n} I({}_nX_{A_i}(s)) \\
\Rightarrow \quad & I({}_nX_{\bigcap_{i=1}^{n} A_i}(s)) = \bigcap_{i=1}^{n} I({}_nX_{A_i}(s))
\end{aligned}
$$

$$\Rightarrow \qquad I_n(\bigcap_{i=1}^{n} A_i) = \bigcap_{i=1}^{n} I_n(A_i).$$

Hence I_n is an interior operator.

For the last part, let $A \in D_n\ (\tau_1)$. Thus A is an open set and hence there is an open sequential set $B(s) = \{B_n\}$ in (X, τ_1) such that $B_n = A$. Now,

$$B(s) \subset {}_nX_A(s) \Rightarrow I(B(s)) \subset I({}_nX_A(s))$$

$$\Rightarrow \qquad B(s) \subset I\,({}_nX_A(s)) \Rightarrow \mathrm{A} \subset I_n\,(A) \subset A$$

$$\Rightarrow \qquad I_n\,(A) = A \in {}_{In}\tau.$$

Again let $A \in I_n\tau$. Therefore, $I_n(A) = A$.

Let $B(s) = I\,({}_nX_A(s))$ then B(s) is an open sequential set and its n^{th} *component is* $I_n\,(A) = A$ [Since, $A \in I_n\tau$]. Therefore, $A \in D_n(\tau_1)$ and the proof is done.

Note 2.1. *The sequential interior operator in Example 2.1 generates an ST on X but it is not discrete though every component of this operator induces discrete topology on X.*

Definition 2.4. Let X be a nonvoid set, F(s) be a sequential set in X and $I : (P(X))^{\mathrm{N}} \to (P(X))^{\mathrm{N}}$ be a sequential interior operator. A function $I_n^{F(s)} : P(X) \to P(X)$ defined by $I_n^{F(s)}\,(A) = P_n\,(I({}_nF_A(s)))$ is called n^{th} relative interior operator of I with respect to $F(s)$.

Note 2.2. *If* $I : (P(X))^{\mathrm{N}} \to (P(X))^{\mathrm{N}}$ *is a sequential interior operator then obviously* $I_n^{X(s)} = I_n$ *and consequently* $I_n^{x(s)\tau} = I_n\,\tau$.

Theorem 2.4. *Let X be a nonvoid set and* $I_n^{F(s)} : P(X) \to P(X)$ *be a relative interior operator of the sequential interior operator* $I : (P(X))^{\mathrm{N}} \to (P(X))^{\mathrm{N}}$ *with respect to sequential set F(s). Then :*

(i) $I_n^{F(s)}$ *is contractive,*

(ii) $I_n^{F(s)}$ *is idempotent,*

(iii) $I_n^{F(s)}$ *commutes with finite intersection,*

(iv) X need not be a fixed point of $I_n^{F(s)}$.

Proof. (i) Let $A \in P(X)$. Then

$$I({}_nF_A(s)) \subset {}_nF_A(s)$$

$$\Rightarrow \quad I_n^{F(s)}(A) \subset A.$$

Therefore $I_n^{F(s)}$ is contractive.

(ii) Let $A \in P(X)$ then $I_n^{F(s)}(A) \subset A$. Therefore

$${}_nF_{I_n^{F(s)}(A)}(s) \subset {}_nF_A(s)$$

$$\Rightarrow I({}_nF_{I_n^{F(s)}(A)}(S)) \subset I({}_nF_A(s))$$

$$\Rightarrow I_n{}^{F(s)}(I_n^{F(s)}(A)) \subset I_n^{F(s)}(A). \rightarrow (1)$$

Also, $I(I({}_nF_A(s)) = I({}_nF_A(s))$. Since

$$I({}_nF_A(s)) \subset {}_nF_{I_n^{F(s)}(A)}(S)$$

$$\Rightarrow I(I({}_nF_A(s))) \subset \mathrm{I}({}_nF_{I_n^{F(s)}(A)}(S))$$

$$\Rightarrow I({}_nF_A(s)) \subset \mathrm{I}({}_nF_{I_n^{F(s)}(A)}(S))$$

$$\Rightarrow I_n^{F(s)}(A) \subset I_n^{F(s)}(I_n^{F(s)}(A)). \rightarrow (2)$$

Combining (1) and (2), we get $I_n^{F(s)}(A) = I_n^{F(s)}(I_n^{F(s)}(A))$. Hence $I_n^{F(s)}$ is idempotent.

(iii) Let $A_i \in P(X)$, i = 1(1)m. Now

$$I\left(\bigcap_{i=1}^{m} {}_nF_{Ai}(s)\right) = \bigcap_{i=1}^{m} I({}_nF_{Ai}(s))$$

$$\Rightarrow I({}_nF_{\bigcap_{i=1}^{m} Ai}(s)) = \bigcap_{i=1}^{m} I({}_nF_{Ai}(s))$$

$$\Rightarrow I_n^{F(s)}(\bigcap_{i=1}^{m} A_i) = (\bigcap_{i=1}^{m} I_n^{F(s)}(A_i).$$

That is, $I_n^{F(s)}$ commutes with finite intersection.

(iv) Let X be a nonvoid set. Define a function $I : (P(X))^N \to (P(X))^N$ by

$$I(A)(s)) = \phi(s), \text{ whenver } A(s) \neq X(s)$$
$$= X(s), \text{ whenver } A(s) = X(s)$$

It is a sequential interior operator and for any sequential set $F(s) \neq X(s)$ in X, $I_n^{F(s)}(X) = \phi$ for all those $n \in N$ for which ${}_nF_X(s) \neq X(s)$.

3. DENSITY PROPERTY OF SEQUENTIAL SETS

$K\Omega$–closure operator [8] expands a sequential set and sequential interior operator contracts. This section mainly deals with two kinds of sequential sets, one is extremely expansive and other is exremely contractive under some $K\Omega$–closure and sequential interior operators. We should mention that some examples used in this section are invited from [10]

Definition 3.1. Let X be a nonvoid set. Seppose $C : (P(X))^N \to (P(X))^N$ be a $K\Omega$–closure operator and $I : (P(X))^N \to (P(X))^N$ be a sequential interior operator, then (X, C, I) is called $K\Omega$–space.

Definition 3.2. Let (X, C, I) be a $K\Omega$–space. The closure operator $C_I : (P(X))^N \to (P(X))^N$ induced by the sequential interior operator $I \cdot (P(X))^N \to (P(X))^N$ is defined by $C_I(A)(s)) = X(s) - I(X(s) - A(s))$ for all $\{A_n\}_{n=1}^{\infty} \in P(X)^N$.

Definition 3.3. Let (X, C, I) be a $K\Omega$-space. The sequential interior operator $I_C : (P(X))^N \to (P(X))^N$ induced by the $K\Omega$-closure operater $C : (P(X))^N \to (P(X))^N$ is defined by $I_C(A(s)) = X(s) - C(X(s) - A(s))$ for all $\{A_n\}_{n=1}^{\infty} \in P(X)^N$.

Definition 3.4. A sequential point $\alpha = (x, P)$ in a $K\Omega$-space (X, C, I) is called I-limit point of a sequential set $A(s) = \{A_n\}_{n=1}^{\infty}$ in X if for any sequential set B(s) such that $\alpha \in_{\omega} I(B(s)) = \{H_n\}_{n=1}^{\infty}$, either

$$(a)\ x \in H_n \cap A_n \text{ for some } n \notin P$$

$$or\quad (b)\ y \in H_n \cap A_n \text{ for some } n \in N \text{ and } y \neq x.$$

Definition 3.5. In a $K\Omega$-space (X, C, I), a sequential set $A(s) = \{A_n\}_{n=1}^{\infty}$ is said to be I-open if $I(A(s)) = A(s)$ and C-open if $I_C(A(s)) = A(s)$.

Note 3.1. *Let D be a topology on a nonvoid set X. Then* $C_D : (P(X))^N \to (P(X))^N$ *defined by* $C_D(A)(s)) = \{cl(A_n)\}_{n=1}^{\infty}$ *for all* $A(s) = \{A_n\}_{n=1}^{\infty} \in (P(X))^N$ *is a* $K\Omega$*-closure operator, where 'cl' denotes the closure operator in (X,D).*

Note 3.2. *Let D be a topology on a nonvoid set X. Then* $I_D : (P(X))^N \to (P(X))^N$ *is a sequential interior operator defined by* $I_D(A(s)) = \{int(A_n)\}_{n=1}^{\infty}$ *for all* $A(s) = \{A_n\}_{n=1}^{\infty} \in (P(X))^N$*, where 'int' denotes the interior operator in (X, D).*

Definition 3.6. A sequential set $A(s)$ in a $K\Omega$-space (X, C, I) is said to be $(\omega)C$–dense if $C(A(s)) = X(s)$. If further $C_n(A_n) = X$ for all $n \in N$ then $A(s)$ is called C–dense in (X, C, I).

Definition 3.7. A sequential set $A(s)$ in a $K\Omega$-space (X, C, I) is said to be $(\omega)I$–dense if $C_I(A(s)) = X(s)$. If further $C_{In}(A_n) = X$ for all $n \in N$ then $A(s)$ is called I–dense in (X, C, I).

Note 3.3. *Let C and I be the closure and interior operators in an STS* (X, τ)*. Since* $\tau_C = \tau_I = \tau$*, the space* (X, τ) *can be identified with the* $K\Omega$*-space* (X, C, I)*.*

Example 3.1. Let $X = \{1,2,3,4\}$ and $\tau = \{\phi(s)$, A(s), B(s), C(s), D(s), E(s), F(s), G(s), H(s), M(s), N(s), P(s), Q(s), R(s), S(s), T(s), X(s)$\}$, where

$$A(s) = \{A_n\}_{n=1}^{\infty};\ A_n = \{1\};\ \text{whenever n is odd,}$$
$$= \phi;\ \text{whenever n is even.}$$

$$
\begin{aligned}
B(s) = \{B_n\}_{n=1}^{\infty};\ B_n &= \{3\};\ \text{whenever n is odd,}\\
&= \phi;\ \text{whenever n is even.}\\
C(s) = \{c_n\}_{n=1}^{\infty};\ C_n &= \phi;\ \text{whenever n is odd,}\\
&= \{1\};\ \text{whenever n is even.}\\
D(s) = \{D_n\}_{n=1}^{\infty};\ D_n &= \phi;\ \text{whenever n is odd,}\\
&= \{2\};\ \text{whenever n is even.}\\
E(s) = \{E_n\}_{n=1}^{\infty};\ E_n &= \{1,3\};\ \text{whenever n is odd,}\\
&= \phi;\ \text{whenever n is even.}\\
F(s) = \{F_n\}_{n=1}^{\infty};\ F_n &= \{1\};\ \text{FOR ALL } n \in N.\\
G(s) = \{G_n\}_{n=1}^{\infty};\ G_n &= \{1\};\ \text{whenever n is odd,}\\
&= \{2\};\ \text{whenever n is even.}\\
H(s) = \{H_n\}_{n=1}^{\infty};\ H_n &= \{3\};\ \text{whenever n is odd,}\\
&= \{1\};\ \text{whenever n is even.}\\
M(s) = \{M_n\}_{n=1}^{\infty};\ M_n &= \{3\};\ \text{whenever n is odd,}\\
&= \{2\};\ \text{whenever n is even.}\\
N(s) = \{N_n\}_{n=1}^{\infty};\ N_n &= \phi;\ \text{whenever n is odd,}\\
&= \{1,2\};\ \text{whenever n is even.}\\
P(s) = \{P_n\}_{n=1}^{\infty};\ P_n &= \{1,3\};\ \text{whenever n is odd,}\\
&= \{1\};\ \text{whenever n is even.}\\
Q(s) = \{Q_n\}_{n=1}^{\infty};\ Q_n &= \{1,3\};\ \text{whenever n is odd,}\\
&= \{2\};\ \text{whenever n is even.}\\
R(s) = \{R_n\}_{n=1}^{\infty};\ R_n &= \{1,3\};\ \text{whenever n is odd,}\\
&= \{1,2\};\ \text{whenever n is even.}
\end{aligned}
$$

$$S(s) = \{S_n\}_{n=1}^{\infty};\ S_n = \{1\};\ \text{whenever n is odd,}$$
$$= \{1,2\};\ \text{whenever n is even.}$$
$$T(s) = \{T_n\}_{n=1}^{\infty};\ T_n = \{3\};\ \text{whenever n is odd,}$$
$$= \{1,2\};\ \text{whenever n is even.}$$

The sequential set $U(s) = \{U_n\}_{n=1}^{\infty}$, where

$$U_n = \{1,2,3\};\ \text{if n is odd,}$$
$$= \{1,2,4\};\ \text{otherwise}$$

is C–dense in the $K\Omega$–space (X, C, I), where C and I are the closure and interior operators in the STS (X, τ).

Theorem 3.1. *Let D be a topology on X. In the $K\Omega$-space (X, C_D, I_D) if a sequential set $A(s) = \{A_n\}_{n=1}^{\infty}$ is $(\omega)C$–dense then it is C-dense.*

Proof. Since A(s) is $(\omega)C$-dense then $C_D(A(s)) = X(s)$. Again from the definition of C_D we have $C_D(A(s)) = \{cl(A_n)\}_{n=1}^{\infty}$. Thus $X(s) = \{cl(A_n)\}_{n=1}^{\infty} \Rightarrow X = cl(A_n)$. Hence the proof is completed.

Note 3.4. *In the $K\Omega$–space (X, C_D, I_D), C–density and I–density are same. For,*

$$\begin{aligned} C_{ID}(A(s)) &= X(s) - I_D(X(s) - A(s)) \\ &= X(s) - I_D\{X - A_n\} \\ &= X(s) - \{int(X - A_n)\} \\ &= \{X - int(X - A_n)\} \\ &= \{cl(A_n)\} \\ &= C_D(A(s)). \end{aligned}$$

Note 3.5. *Let $K\Omega$–closure operator C and sequential interior operator I be connected by $C(A)(s)) = X(s) - I(X(s) - A(s))$, for any sequential set $A(s)$ in the underlying set X. Then C-density and I-density in the corresponding $K\Omega$–space (X, C, I) coincide.*

Definiton 3.8. A sequential set $A(s)$ in $K\Omega$–space (X, C, I) is said to be (ω)CI–nowhere dense if $I(C(A(s))) = \phi(s)$. If further $I_n(C_n(A_n)) = \phi$ then $A(s)$ is called CI–nowhere dense.

Example 3.2. In Example 3.1 the sequential set $V(s) = \{V_n\}_{n=1}^{\infty}$, where

$$V_n = \{2\};\ \text{when n is odd}$$
$$= \{3\};\ \text{otherwise}$$

is CI–nowhere dense. Because, $C(V(s)) = \{2,4\}, \{3,4\}, \{2,4\}, \{3,4\}$........ and so, $IC(V(s)) = \phi\,(s)$ along with $I_n(C_n(V_n)) = \phi$.

Note 3.6. *From the Definitions 3.6 and 3.8 we observe that C–dense and CI-nowhere dence sequential sets are also (ω) C-dense and (ω) CI–nowhere dense sequential sets respectively. But the converses need not be true as seen from the following example.*

Example 3.3. Consider $X = \{1,2,3,4\}$ and $\tau = \{\phi(s), A(s), B(s), C(s), D(s), X(s)\}$, where

$$A(s) = \{A_n\}_{n=1}^{\infty};\ A_n = \{1\};\ \text{n is odd,}$$
$$= \phi;\ \text{n is even.}$$

$$B(s) = \{B_n\}_{n=1}^{\infty};\ B_n = \{1,3\};\ \text{n is odd,}$$
$$= X;\ \text{n is even.}$$

$$C(s) = \{c_n\}_{n=1}^{\infty};\ C_n = \{3\};\ \text{n is odd,}$$
$$= \phi;\ \text{n is even.}$$

$$D(s) = \{D_n\}_{n=1}^{\infty};\ D_n = \{1,3\};\ \text{n is odd,}$$
$$= \phi;\ \text{n is even.}$$

Then the sequential sets $U(s) = \{U_n\}_{n=1}^{\infty}$ and $V(s) = \{V_n\}_{n=1}^{\infty}$ defined by,

$$U_n = \{1,3\};\ \text{n is odd,}$$
$$= \phi,\ \text{n is even.}$$

and

$$V_n = \{2\};\ \text{n is odd,}$$
$$= X,\ \text{n is even.}$$

are $(\omega)C$–dense and $(\omega)CI$-nowhere dense respectively in the $K\Omega$-*space* (X, C, I), where C and I are the closure and interior operators in the STS (X, τ). But neither $U(s)$ is C–dense nor $V(s)$ is CI-nowhere dense in (X, τ).

Theorem 3.2. *Let D be a topology on X. In the $K\Omega$-space (X, C_D, I_D) if a sequential set $A(s) = \{A_n\}_{n=1}^{\infty}$ is $(\omega)C_DI_D$-nowhere dense then it is C_DI_D-nowhere dense.*

Proof. Proof is omitted.

Theorem 3.3. *Let $A(s)$ be a sequential set in a $K\Omega$-space (X, C, I) so that $C(I(A(s))) \subset C_I(I(A(s)))$ and $I_C(I(A(s))) = I(A(s))$* and *$I_C(I(A(s)) = I(A(s))$. Then $C(I(A(s))) - I(A(s))$ is $(\omega)CI$-nowhere dense.*

Proof. Assume that $I(C(C(I(A(s))) - I(A(s))) \neq \phi\ (s)$. So there exist at least one sequential point $\alpha = (x, P)$ such that $\alpha = (x, P) \in I(C(C(I(A(s))) - I(A(s))))$. As $I(C(C(I(A(s))) - I(A(s))))$ is I-open, there exists a I-open sequential set $G(s) = \{G_n\}_{n=1}^{\infty}$ such that,

$$\begin{aligned} \alpha \in G(s) &\subset I(C(C(I(A(s))) - I(A(s)))) \\ &\subset C(C(I(A(s)))) - I(A(s))[\text{since I is contractive}] \\ &= C(I(A(s))) - I(A(s))[\text{since I(A(s)) is C-open}] \\ &\subset C_I(I(A(s)))) - I(A(s)). \end{aligned}$$

We claim that none of the sequential points belonging to $I(A(s))$ is an I-limit point of $C_I(I(A(s))) - I(A(s))$. If $\beta = (y, Q) \in I(A(s))$, then there exists an I-open sequential set $G_I(s)$ such that $\beta = (y, Q) \in G_I(s) \subset I(A(s))$. Since $G_I(s)$ is an I-nbd of β, it is also a *I–weak* nbd of β with $G_I(s) \cap (C_I(I(A(s))) - I(A(s))) = \phi(s)$. So $\beta \notin_\omega C_I(I(A(s))) - I(A(s))$. Since $G(s) \subset C_I(I(A(s))) - I(A(s)) \Rightarrow \beta \notin_\omega G(s)$. Therefore we have $I(A(s)) \subset G^c(s)$, this implies $C_I(I(A(s))) \subset C_I(G^c(s)) = G^c(s)$. Thus we get $G(s) \subset C_I(I(A(s))) - I(A(s)) \subset G^c(s) - I(A(s))$, which is a contradiction. Therefore, $C(I(A(s))) - I(A(s))$ is $(\omega)CI$-nowhere dense.

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